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ON ALMOST CONTRA $e^*\theta$-CONTINUOUS FUNCTIONS

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Abstract. The aim of this paper is to introduce and investigate some of fundamental properties of almost contra $e^*\theta$-continuous functions via $e^*\theta$-closed sets which are defined by Farhan and Yang [15]. Also, we obtain several characterizations of almost contra $e^*\theta$-continuous functions. Furthermore, we investigate the relationships between almost contra $e^*\theta$-continuous functions and seperation axioms and $e^*\theta$-closedness of graphs of functions.

1. Introduction

In 2006, the concept of almost contra continuity [4], which is stronger than almost contra precontinuity [8] is introduced by Ekici and almost contra β-continuity [4] introduced by Baker, is defined. In 2017, some properties and characterizations of the notion of almost contra $\beta\theta$-continuous function [5] defined by Caldas via $\beta\theta$-closed sets are obtained. The notion of almost contra $e^*\theta$-continuity is stronger than almost contra e^*-continuity which is defined by us in this manuscript. In this paper, we introduce some new forms of contra e^*-continuity [9] defined by Ekici. Also, we obtain some characterizations of almost contra $e^*\theta$-continuous functions and investigate their some fundamental properties. Moreover, we investigate the relationships between almost contra $e^*\theta$-continuity and other related generalized forms of contra continuity.

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2. Preliminaries

Throughout this present paper, X and Y represent topological spaces. For a subset A of a space X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively. The family of all closed (resp. open) sets of X is denoted by $C(X)$(resp. $O(X)$). A subset A is said to be regular open [28] (resp. regular closed [28]) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). A point $x \in X$ is said to be δ-cluster point [30] of A if $int(cl(U)) \cap A \neq \emptyset$ for each open neighbourhood U of x. The set of all δ-cluster points of A is called the δ-closure [30] of A and is denoted by $cl_\delta(A)$. If $A = cl_\delta(A)$, then A is called δ-closed [30], and the complement of a δ-closed set is called δ-open [30]. The set $\{x|(\exists U \in \tau)(x \in U)(int(cl(U)) \subseteq A)\}$ is called the δ-interior of A and is denoted by $int_\delta(A)$.

A subset A is called α-open [19] (resp. semiopen [17], δ-semiopen [23], preopen [18], δ-preopen [24], b-open [1], e-open [11], e^*-open [12], a-open [10]) if $A \subseteq int(cl(int(A)))$ (resp. $A \subseteq cl(int(A))$, $A \subseteq cl(int_\delta(A))$, $A \subseteq int(cl(A))$, $A \subseteq int(cl_\delta(A))$, $A \subseteq cl(int(A)) \cup int(cl(A))$, $A \subseteq cl(int_\delta(A)) \cup int(cl_\delta(A))$, $A \subseteq cl(int(cl_\delta(A)))$, $A \subseteq int(cl(int_\delta(A)))$). The complement of an α-open (resp. semiopen, δ-semiopen, preopen, δ-preopen, b-open, e-open, e^*-open, a-open) set is called α-closed [19] (resp. semiclosed [17], δ-semiclosed [23], preclosed [18], δ-preclosed [24], b-closed [1], e-closed [11], e^*-closed [12], a-closed [10]). The intersection of all e^*-closed (resp. a-closed, semiclosed, δ-semiclosed, preclosed, δ-preclosed) sets of X containing A is called the e^*-closure [12] (resp. a-closure [10], semiclosure [17], δ-semiclosure [23], preclosure [18], δ-preclosure [24]) of A and is denoted by e^*-$cl(A)$ (resp. a-$cl(A)$, $scl(A)$, δ-$scl(A)$, $pcl(A)$, δ-$pcl(A)$). The union of all e^*-open (resp. a-open, semiopen, δ-semiopen, preopen, δ-preopen) sets of X contained in A is called the e^*-interior [12] (resp. a-interior [10], semiinterior [17], δ-semiinterior [23], preinterior [18], δ-preinterior [24]) of A and is denoted by e^*-$int(A)$ (resp. a-$int(A)$, $sint(A)$, δ-$sint(A)$, $pint(A)$, δ-$pint(A)$).

A point x of X is called a θ-cluster [30] point of A if $cl(U) \cap A \neq \emptyset$ for every open set U of X containing x. The set of all θ-cluster points of A is called the θ-closure [30] of A and is denoted by $cl_\theta(A)$. A subset A is said to be θ-closed [30] if $A = cl_\theta(A)$. The complement of a θ-closed set is called a θ-open [30] set. A point x of X said to be a θ-interior [30] point of a subset A, denoted by $int_\theta(A)$, if there exists an open set U of X containing x such that $cl(U) \subseteq A$.

A point $x \in X$ is said to be a θ-semicluster point [16] of a subset S of X if $cl(U) \cap A \neq \emptyset$ for every semiopen U containing x. The set of all θ-semicluster points of A is called the θ-semiclosure of A and is denoted by θ-$scl(A)$. A subset A is called θ-semiclosed [16] if $A = \theta$-$scl(A)$. The complement of a θ-semiclosed set is called θ-semiopen.

The union of all e^*-open sets of X contained in A is called the e^*-interior [12] of A and is denoted by e^*-$int(A)$. A subset A is said to be e^*-regular [15] if it is e^*-open and e^*-closed. The family of all e^*-regular subsets of X is denoted by $e^*R(X)$.

A point x of X is called an e^*-θ-cluster point of A if e^*-$cl(U) \cap A \neq \emptyset$ for every e^*-open set U containing x. The set of all e^*-θ-cluster points of A is called the e^*-θ-closure [15] of A and is denoted by e^*-$cl_\theta(A)$. A subset A is said to be e^*-θ-closed if $A = e^*$-$cl_\theta(A)$. The complement of an e^*-θ-closed set is called an e^*-θ-open [15] set. A point x of X said to be an e^*-θ-interior [15] point of a subset A, denoted by e^*-$int_\theta(A)$, if there exists an e^*-open set U of X containing x such that e^*-$cl(U) \subseteq A$. Also it is noted in [15] that

$$e^*\text{-regular} \Rightarrow e^*\text{-}\theta\text{-open} \Rightarrow e^*\text{-open}.$$

The family of all e^*-θ-open (resp. e^*-θ-closed, e^*-open, e^*-closed, regular open, regular closed, δ-open, δ-closed, θ-open, θ-closed, θ-semiopen, θ-semiclosed, semiopen, semiclosed, preopen, preclosed, δ-semiopen, δ-semiclosed, δ-preopen, δ-preclosed, a-open, a-closed) subsets of X is denoted by $e^*\theta O(X)$ (resp. $e^*\theta C(X)$, $e^*O(X)$, $e^*C(X)$, $RO(X)$, $RC(X)$, $\delta O(X)$, $\delta C(X)$, $\theta O(X)$, $\theta C(X)$, $\theta SO(X)$, $\theta SC(X)$, $SO(X)$, $SC(X)$,

$PO(X)$, $PC(X)$, $\delta SO(X)$, $\delta SC(X)$, $\delta PO(X)$, $\delta PC(X)$), $aO(X)$, $aC(X)$). The family of all open (resp. closed, e^*-θ-open, e^*-θ-closed, e^*-open, e^*-closed, regular open, regular closed, δ-open, δ-closed, θ-open, θ-closed, θ-semiopen, θ-semiclosed, semiopen, semiclosed, preopen, preclosed, δ-semiopen, δ-semiclosed, δ-preopen, δ-preclosed, a-open, a-closed) sets of X containing a point x of X is denoted by $O(X,x)$ (resp. $C(X,x)$, $e^*\theta O(X,x)$, $e^*\theta C(X,x)$, $e^*O(X,x)$, $e^*C(X,x)$, $RO(X,x)$, $RC(X,x)$, $\delta O(X,x)$, $\delta C(X,x)$, $\theta O(X,x)$, $\theta C(X,x)$, $\theta SO(X,x)$, $\theta SC(X,x)$, $SO(X,x)$, $SC(X,x)$, $PO(X,x)$, $PC(X,x)$, $\delta SO(X,x)$, $\delta SC(X,x)$, $\delta PO(X,x)$, $\delta PC(X,x)$, $aO(X,x)$, $aC(X,x)$).

We shall use the well-known accepted language almost in the whole of the proofs of the theorems in this article.

Lemma 2.1. [12] *Let A be a subset of a space X, then the followings hold:*
(1) e^*-$cl(X \setminus A) = X \setminus e^*$-$int(A)$,
(2) $x \in e^*$-$cl(A)$ *if and only if* $A \cap U \neq \emptyset$ *for every* $U \in e^*O(X,x)$,
(3) A *is* $e^*C(X)$ *if and only if* $A = e^*$-$cl(A)$,
(4) e^*-$cl(A) \in e^*C(X)$,
(5) e^*-$int(A) = A \cap cl(int(cl_\delta(A)))$.

Lemma 2.2. [10, 23, 24] *Let A be a subset of a space X, then the followings hold:*
(1) a-$cl(A) = A \cup cl(int(cl_\delta(A)))$,
(2) δ-$scl(A) = A \cup int(cl_\delta(A))$,
(3) δ-$pcl(A) = A \cup cl(int_\delta(A))$.

Lemma 2.3. [15] *The following properties hold for the $e^*\theta$-closure of a subset A of a topological space X.*
(1) $A \subseteq e^*$-$cl(A) \subseteq e^*$-$cl_\theta(A)$,
(2) If $A \in e^*\theta O(X)$, *then* e^*-$cl_\theta(A) = e^*$-$cl(A)$,
(3) If $A \subseteq B$, *then* e^*-$cl_\theta(A) \subseteq e^*$-$cl_\theta(B)$,

(4) $e^*\text{-}cl_\theta(A) \in e^*\theta C(X)$ *and* $e^*\text{-}cl_\theta(e^*\text{-}cl_\theta(A)) = e^*\text{-}cl_\theta(A)$,

(5) If $A_\alpha \in e^*\theta C(X)$ *for each* $\alpha \in \Lambda$, *then* $\cap\{A_\alpha | \alpha \in \Lambda\} \in e^*\theta C(X)$,

(6) If $A_\alpha \in e^*\theta O(X)$ *for each* $\alpha \in \Lambda$, *then* $\cup\{A_\alpha | \alpha \in \Lambda\} \in e^*\theta O(X)$,

(7) $e^*\text{-}cl_\theta(X \setminus A) = X \setminus e^*\text{-}int_\theta(A)$.

Lemma 2.4. [15] *Let A be a subset of a topological space X, then the followings hold:*

(1) If $A \in e^*O(X)$, *then* $e^*\text{-}cl_\theta(A) \in e^*R(X)$,

(2) $A \in e^*R(X)$ *if and only if* $A \in e^*\theta O(X) \cap e^*\theta C(X)$,

(3) A is $e^*\theta$-open in X if and only if for each $x \in A$ there exists $U \in e^*R(X, x)$ such that $x \in U \subseteq A$.

Definition 2.1. Let A be a subset of a space X. The intersection of all regular open sets in X containing A is called the r-kernel of A [9] and is denoted by $rker(A)$.

Lemma 2.5. [9] *The following properties hold for subsets A and B of a space X.*

(1) $x \in rker(A)$ *if and only if* $A \cap F \neq \emptyset$ *for any* $F \in RC(X, x)$,

(2) $A \subseteq rker(A)$,

(3) If A is regular open in X, then $A = rker(A)$,

(4) If $A \subseteq B$, *then* $rker(A) \subseteq rker(B)$.

Lemma 2.6. [11] *The following properties hold for a subset A of a space X.*

(1) $cl(int_\delta(A)) = cl_\delta(int_\delta(A))$,

(2) $int(cl_\delta(A)) = int_\delta(cl_\delta(A))$.

Lemma 2.7. *Let A be a subset of a topological space X. If A is an e^*-open set in X, then* $int_\delta(X \setminus A) = X \setminus cl_\delta(A) \in RO(X)$.

Proof. Let $A \in e^*O(X)$.

$$\begin{array}{lll} A \in e^*O(X) & \Rightarrow & A \subseteq cl(int(cl_\delta(A))) \\ & \Rightarrow & cl_\delta(A) \subseteq cl_\delta(cl(int(cl_\delta(A)))) \overset{\text{Lemma 2.6}}{=} cl_\delta(cl_\delta(int_\delta(cl_\delta(A)))) \\ & \Rightarrow & cl_\delta(A) \subseteq cl_\delta(cl(int(cl_\delta(A)))) = cl_\delta(int_\delta(cl_\delta(A))) \\ & \Rightarrow & cl_\delta(A) \subseteq cl_\delta(cl(int(cl_\delta(A)))) \overset{\text{Lemma 2.6}}{=} cl(int(cl_\delta(A))) \\ & \Rightarrow & \setminus cl(int(cl_\delta(A))) = int(cl(\setminus cl_\delta(A))) \subseteq \setminus cl_\delta(A) \ldots (*) \end{array}$$

$$\begin{array}{lll} int(cl_\delta(A)) \subseteq cl_\delta(A) & \Rightarrow & cl(int(cl_\delta(A))) = cl_\delta(int(cl_\delta(A))) \subseteq cl_\delta(cl_\delta(A)) = cl_\delta(A) \\ & \Rightarrow & \setminus cl_\delta(A) \subseteq \setminus cl(int(cl_\delta(A))) = int(cl(\setminus cl_\delta(A))) \ldots (**) \end{array}$$

$$(*), (**) \Rightarrow \setminus cl_\delta(A) = int(cl(\setminus cl_\delta(A))) \Rightarrow \setminus cl_\delta(A) \in RO(X).$$ □

Definition 2.2. A function $f : X \to Y$ is said to be:

a) $e^*\theta$-continuous (briefly $e^*\theta$.c.) if $f^{-1}[V]$ is e^*-θ-closed in X for every $V \in C(Y)$,

b) almost $e^*\theta$-continuous (briefly a.$e^*\theta$.c.) if $f^{-1}[V]$ is e^*-θ-closed in X for every regular closed set V in Y,

c) contra R-map [9] (resp. contra continuous [7], contra $e^*\theta$-continuous [3], contra e^*-continuous [13]) if $f^{-1}[V]$ is regular closed (resp. closed, e^*-θ-closed, e^*-closed) in X for every regular open (resp. open, open, open) set V in Y,

d) almost contra precontinuous [8] (resp. almost contra continuous [4], almost contra β-continuous [4], almost contra e^*-continuous) if $f^{-1}[V]$ is preclosed (resp. closed, β-closed, e^*-closed) in X for every regular open set V in Y.

Lemma 2.8. [25] *For a topological space (X, τ) the followings are equivalent:*

(1) (X, τ) is almost regular;

(2) For each point $x \in X$ and each neighbourhood M of x, there exists a regular open neighbourhood V of x such that $cl(V) \subseteq int(cl(M))$.

3. Almost Contra $e^*\theta$-continuous Functions

Definition 3.1. A function $f : X \to Y$ is said to be almost contra $e^*\theta$-continuous (briefly a.c.$e^*\theta$.c.) if $f^{-1}[V]$ is e^*-θ-closed in X for each regular open set V of Y.

Theorem 3.1. *For a function $f : X \to Y$, the following properties are equivalent:*
(1) f is almost contra $e^*\theta$-continuous;
(2) The inverse image of each regular closed set in Y is e^*-θ-open in X;
(3) For each point $x \in X$ and each $V \in RC(Y, f(x))$, there exists $U \in e^*\theta O(X, x)$ such that $f[U] \subseteq V$;
(4) For each point $x \in X$ and each $V \in SO(Y, f(x))$, there exists $U \in e^*\theta O(X, x)$ such that $f[U] \subseteq cl(V)$;
(5) $f[e^*\text{-}cl_\theta(A)] \subseteq rker(f[A])$ for every subset A of X;
(6) $e^*\text{-}cl_\theta(f^{-1}[B]) \subseteq f^{-1}[rker(B)]$ for every subset B of Y;
(7) $f^{-1}[cl_\delta(V)]$ is e^*-θ-open for every $V \in e^*O(Y)$;
(8) $f^{-1}[cl_\delta(V)]$ is e^*-θ-open for every $V \in \delta SO(Y)$;
(9) $f^{-1}[int(cl_\delta(V))]$ is e^*-θ-closed for every $V \in \delta PO(Y)$;
(10) $f^{-1}[int(cl_\delta(V))]$ is e^*-θ-closed for every $V \in O(Y)$;
(11) $f^{-1}[cl(int_\delta(V))]$ is e^*-θ-open for every $V \in C(Y)$.

Proof. (1) $\Rightarrow$ (2) : Let $V \in RC(Y)$.

$$\left.\begin{array}{r} V \in RC(Y) \Rightarrow \setminus V \in RO(Y) \\ (1) \end{array}\right\} \Rightarrow \setminus f^{-1}[V] = f^{-1}[\setminus V] \in e^*\theta C(X)$$

$\Rightarrow f^{-1}[V] \in e^*\theta O(X)$.

(2) $\Rightarrow$ (3) : Let $x \in X$ and $V \in RC(Y, f(x))$.

$$\left.\begin{array}{r} (x \in X)(V \in RC(Y, f(x))) \\ (2) \end{array}\right\} \Rightarrow (U := f^{-1}[V] \in e^*\theta O(X, x))(f[U] \subseteq V).$$

(3) $\Rightarrow$ (4) : Let $x \in X$ and $V \in SO(Y, f(x))$.

$$\left.\begin{array}{r} V\in SO(Y,f(x))\Rightarrow cl(int(V))\in RC(Y,f(x)) \\ (3) \end{array}\right\}\Rightarrow$$

$\Rightarrow (\exists U\in e^{*}\theta O(X,x))(f\left[U\right]\subseteq cl(int(V))\subseteq cl(V))$.

$(4)\Rightarrow(5)$: Let $A\subseteq X$ and $x\notin f^{-1}[rker(f[A])]$.

$x\notin f^{-1}[rker(f[A])]\Rightarrow f(x)\notin rker(f[A])\Rightarrow(\exists F\in RC(Y,f(x)))(F\cap f[A]=\emptyset)$

$$\left.\begin{array}{r} \Rightarrow(\exists F\in SO(Y,f(x)))(f^{-1}[F]\cap A=\emptyset) \\ (4) \end{array}\right\}\Rightarrow$$

$\Rightarrow(\exists U\in e^{*}\theta O(X,x))(f[U]\subseteq cl(F)=F)(f^{-1}[F]\cap A=\emptyset)$

$\Rightarrow(\exists U\in e^{*}\theta O(X,x))(U\subseteq f^{-1}[F])(f^{-1}[F]\cap A=\emptyset)$

$\Rightarrow(\exists U\in e^{*}\theta O(X,x))(U\cap A=\emptyset)$

$\Rightarrow x\notin e^{*}\text{-}cl_{\theta}(A)$.

$(5)\Rightarrow(6)$: Let $B\subseteq Y$.

$$\left.\begin{array}{r} B\subseteq Y\Rightarrow f^{-1}[B]\subseteq X \\ (5) \end{array}\right\}\Rightarrow f[e^{*}\text{-}cl_{\theta}(f^{-1}\left[B\right])]\subseteq rker(f\left[f^{-1}\left[B\right]\right])\subseteq rker(B)$$

$\Rightarrow e^{*}\text{-}cl_{\theta}(f^{-1}\left[B\right])\subseteq f^{-1}\left[rker(B)\right]$.

$(6)\Rightarrow(7)$: Let $V\in e^{*}O(Y)$.

$$\left.\begin{array}{r} V\in e^{*}O(Y)\overset{\text{Lemma 2.7}}{\Rightarrow}\setminus cl_{\delta}(V)\in RO(Y) \\ (6) \end{array}\right\}\Rightarrow$$

$\Rightarrow e^{*}\text{-}cl_{\theta}(f^{-1}\left[\setminus cl_{\delta}(V)\right])\subseteq f^{-1}\left[rker(\setminus cl_{\delta}(V))\right]=f^{-1}\left[\setminus cl_{\delta}(V)\right]$

$\Rightarrow\setminus e^{*}\text{-}int_{\theta}(f^{-1}\left[cl_{\delta}(V)\right])\subseteq\setminus f^{-1}\left[cl_{\delta}(V)\right]$

$\Rightarrow f^{-1}\left[cl_{\delta}(V)\right]\subseteq e^{*}\text{-}int_{\theta}(f^{-1}\left[cl_{\delta}(V)\right])$

$\Rightarrow f^{-1}\left[cl_{\delta}(V)\right]\in e^{*}\theta O(X)$.

$(7)\Rightarrow(8)$: This is obvious since every δ-semiopen set is e^{*}-open.

$(8)\Rightarrow(9)$: Let $V\in\delta PO(Y)$.

$$\left.\begin{array}{r} V\in\delta PO(Y)\Rightarrow int_{\delta}(\setminus V)\in\delta SO(Y) \\ (8) \end{array}\right\}\Rightarrow f^{-1}\left[cl_{\delta}(int_{\delta}(\setminus V))\right]\in e^{*}\theta O(X)$$

$\Rightarrow \backslash f^{-1}\left[int_\delta(cl_\delta(V))\right] \in e^*\theta O(X)$

$\Rightarrow f^{-1}\left[int(cl_\delta(V))\right] \in e^*\theta C(X)$.

(9) $\Rightarrow$ (10) : This is obvious since every open set is δ-preopen.

(10) $\Rightarrow$ (11) : Clear.

(11) $\Rightarrow$ (1) : Let $V \in RO(Y)$.

$$\left.\begin{array}{r} V \in RO(Y) \Rightarrow (V = int(cl_\delta(V)))(\backslash V \in C(Y)) \\ (11) \end{array}\right\} \Rightarrow$$

$\Rightarrow f^{-1}[\backslash V] = \backslash f^{-1}[V] = \backslash f^{-1}\left[int(cl_\delta(V))\right] = f^{-1}\left[cl(int_\delta(\backslash V))\right] \in e^*\theta O(X)$

$\Rightarrow f^{-1}[V] \in e^*\theta C(X)$. □

Lemma 3.1. *For a subset A of a topological space X, the following properties hold:*

(1) If $A \in e^*O(X)$, then a-$cl(A) = cl_\delta(A)$,

(2) If $A \in \delta SO(X)$, then δ-$pcl(A) = cl_\delta(A)$,

(3) If $A \in \delta PO(X)$, then δ-$scl(A) = int(cl_\delta(A))$,

(4) If $A \in PO(X)$, then $scl(A) = int(cl(A))$.

Proof. (1) Let $A \in e^*O(X)$.

$$\begin{array}{rcl} A \in e^*O(X) & \Rightarrow & A \subseteq cl(int(cl_\delta(A))) \\ & \Rightarrow & cl_\delta(A) \subseteq cl_\delta(cl(int(cl_\delta(A)))) = cl(int(cl_\delta(A))) \\ & \Rightarrow & A \cup cl_\delta(A) = cl_\delta(A) \subseteq A \cup cl(int(cl_\delta(A))) = a\text{-}cl(A) \ldots (*) \end{array}$$

$\delta C(X) \subseteq aC(X) \Rightarrow a$-$cl(A) \subseteq cl_\delta(A) \ldots (**)$

$(*), (**) \Rightarrow a$-$cl(A) = cl_\delta(A)$.

(2) Let $A \in \delta SO(X)$.

$A \in \delta SO(X) \Rightarrow A \subseteq cl(int_\delta(A)) \overset{\text{Lemma 2.6}}{=} cl_\delta(int_\delta(A))$

$$\left.\begin{array}{r} \Rightarrow cl_\delta(A) \subseteq cl_\delta(cl_\delta(int_\delta(A))) = cl_\delta(int_\delta(A)) = cl(int_\delta(A)) \\ \delta\text{-}pcl(A) = A \cup cl(int_\delta(A)) \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow \delta\text{-}pcl(A) \supseteq A \cup cl_\delta(A) = cl_\delta(A) \\ \delta C(X) \subseteq \delta PC(X) \Rightarrow \delta\text{-}pcl(A) \subseteq cl_\delta(A) \end{array}\right\} \Rightarrow \delta\text{-}pcl(A) = cl_\delta(A).$$

(3) Let $A \in \delta PO(X)$.

$$\left.\begin{array}{r} A \in \delta PO(X) \Rightarrow A \subseteq int(cl_\delta(A)) \\ \delta\text{-}scl(A) = A \cup int(cl_\delta(A)) \end{array}\right\} \Rightarrow \delta\text{-}scl(A) = int(cl_\delta(A)).$$

(4) [20]. □

Corollary 3.1. *For a function $f : X \to Y$, the following properties are equivalent:*

(1) f is almost contra $e^*\theta$-continuous;

(2) $f^{-1}[a\text{-}cl(A)]$ is e^*-θ-open for every $A \in e^*O(Y)$;

(3) $f^{-1}[\delta\text{-}pcl(A)]$ is e^*-θ-open for every $A \in \delta SO(Y)$;

(4) $f^{-1}[\delta\text{-}scl(A)]$ is e^*-θ-closed for every $A \in \delta PO(Y)$.

Proof. It follows from Lemma 3.1. □

Theorem 3.2. *For a function $f : X \to Y$, the following properties are equivalent:*

(1) f is almost contra $e^*\theta$-continuous;

(2) $f^{-1}[V]$ is e^*-θ-open in X for each θ-semiopen set of Y;

(3) $f^{-1}[V]$ is e^*-θ-closed in X for each θ-semiclosed set of Y;

(4) $f^{-1}[V] \subseteq e^*\text{-}int_\theta(f^{-1}[cl(V)])$ for every $V \in SO(Y)$;

(5) $f[e^*\text{-}cl_\theta(A)] \subseteq \theta\text{-}scl(f[A])$ for every subset A of X;

(6) $e^*\text{-}cl_\theta(f^{-1}[B]) \subseteq f^{-1}[\theta\text{-}scl(B)]$ for every subset B of Y;

(7) $e^*\text{-}cl_\theta(f^{-1}[V]) \subseteq f^{-1}[\theta\text{-}scl(V)]$ for every open subset V of Y;

(8) $e^*\text{-}cl_\theta(f^{-1}[V]) \subseteq f^{-1}[scl(V)]$ for every open subset V of Y;

(9) $e^*\text{-}cl_\theta(f^{-1}[V]) \subseteq f^{-1}[int(cl(V))]$ for every open subset V of Y.

Proof. (1) ⇒ (2) : Let $V \in \theta SO(Y)$.

$$\left.\begin{array}{r} V \in \theta SO(Y) \Rightarrow (\exists \mathcal{A} \subseteq RC(Y))(V = \cup\mathcal{A}) \\ (1) \end{array}\right\} \Rightarrow$$

$$\Rightarrow f^{-1}[V] = \cup\{f^{-1}[A] \,|\, A \in \mathcal{A}\} \in e^*\theta O(X).$$

(2) ⇒ (3) : Obvious.

(3) ⇒ (4) : Let $V \in SO(Y)$.

$$\left.\begin{array}{r} V \in SO(Y) \Rightarrow \backslash cl(V) \in \theta SC(Y) \\ (3) \end{array}\right\} \Rightarrow$$

$$\Rightarrow f^{-1}[\backslash cl(V)] \in e^*\theta C(X) \Rightarrow \backslash f^{-1}[cl(V)] \in e^*\theta C(X)$$

$$\Rightarrow f^{-1}[cl(V)] \in e^*\theta O(X) \Rightarrow f^{-1}[V] \subseteq f^{-1}[cl(V)] = e^*\text{-}int_\theta(f^{-1}[cl(V)]).$$

$(4) \Rightarrow (5)$: Let $A \subseteq X$ and $x \notin f^{-1}[\theta\text{-}scl(f[A])]$.

$$x \notin f^{-1}[\theta\text{-}scl(f[A])] \Rightarrow f(x) \notin \theta\text{-}scl(f[A]) \Rightarrow (\exists U \in SO(Y, f(x)))(cl(U) \cap f[A] = \emptyset)$$

$$\Rightarrow (\exists U \in SO(Y, f(x)))(f^{-1}[cl(U)] \cap A = \emptyset)$$

$$\left.\begin{array}{r} \Rightarrow (\exists U \in SO(Y, f(x)))(e^*\text{-}int_\theta(f^{-1}[cl(U)]) \cap A = \emptyset) \\ V := e^*\text{-}int_\theta(f^{-1}[cl(U)]) \end{array}\right\} \overset{(4)}{\Rightarrow}$$

$$\Rightarrow (\exists V \in e^*\theta O(X, x))(V \cap A = \emptyset)$$

$$\Rightarrow x \notin e^*\text{-}cl_\theta(A).$$

$(5) \Rightarrow (6)$: Let $B \subseteq Y$.

$$\left.\begin{array}{r} B \subseteq Y \Rightarrow f^{-1}[B] \subseteq X \\ (5) \end{array}\right\} \Rightarrow f[e^*\text{-}cl_\theta(f^{-1}[B])] \subseteq \theta\text{-}scl(f[f^{-1}[B]]) \subseteq \theta\text{-}scl(B)$$

$$\Rightarrow e^*\text{-}cl_\theta(f^{-1}[B]) \subseteq f^{-1}[\theta\text{-}scl(B)].$$

$(6) \Rightarrow (7)$: Obvious.

$(7) \Rightarrow (8)$: This is obvious since $\theta\text{-}scl(V) = scl(V)$ for an open set V.

$(8) \Rightarrow (9)$: Obvious from Lemma 3.1(4).

$(9) \Rightarrow (1)$: Let $V \in RO(Y)$.

$$\left.\begin{array}{r} V \in RO(Y) \subseteq O(Y) \\ (9) \end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(f^{-1}[V]) \subseteq f^{-1}[in(cl(V))] = f^{-1}[V]$$

$$\Rightarrow f^{-1}[V] \in e^*\theta C(X).$$ □

We recall that a topological space X is said to be extremally disconnected if the closure of every open set of X is open in X.

Lemma 3.2. *Let X be a topological space. If X is an extremally disconnected space, then $RO(X) = RC(X)$.*

Theorem 3.3. *Let $f : X \to Y$ be a function. If Y is extremally disconnected, then the following properties are equivalent:*
(1) f is almost contra e$^*\theta$-continuous;
(2) f is almost e$^*\theta$-continuous.

Proof. The proof is obvious from Lemma 3.2. □

Remark 1. From Definitions 2.2 and 3.1, we have the following diagram:

$$\begin{array}{ccccc}
\text{contra } e^*\theta\text{-con.} & \rightarrow & \text{contra } e^*\text{-con.} & \rightarrow & \text{almost contra } e^*\text{-con.} \\
 & \searrow & & \nearrow & \uparrow \\
 & & \text{almost contra } e^*\theta\text{-con.} & & \text{almost contra } \beta\text{-con.} \\
 & & & & \uparrow \\
\text{contra } R\text{-map} & \rightarrow & \text{almost contra con.} & \rightarrow & \text{almost contra pre-con.}
\end{array}$$

Example 3.1. *Let $X := \{a, b, c, d\}$ and $\tau := \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. It is not difficult to see e$^*\theta O(X) = e^*O(X) = 2^X \setminus \{\{c\}, \{d\}, \{c, d\}\}$. Then the identity function $f : (X, \tau) \to (X, \tau)$ is almost contra e$^*\theta$-continuous and so almost contra e*-continuous but f is neither contra e$^*\theta$-continuous nor contra e*-continuous.*

Example 3.2. *Let $X := \{a, b, c, d\}$ and $\tau := \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}\}$. It is not difficult to see e$^*\theta O(X) = e^*O(X) = 2^X \setminus \{\{d\}\}$ and $\beta O(X) = 2^X \setminus \{\{c\}, \{d\}, \{b, c\}, \{c, d\}, \{b, c, d\}\}$. Define the function $f : (X, \tau) \to (X, \tau)$ by $f = \{(a, b), (b, a), (c, c), (d, d)\}$. Then f is almost contra e$^*\theta$-continuous but it is not almost contra β-continuous.*

Theorem 3.4. *If $f : X \to Y$ is an almost contra e$^*\theta$-continuous function which satisfies the property e*-$int_\theta(f^{-1}[cl_\delta(V)]) \subseteq f^{-1}[V]$ for each open set V of Y, then f is e$^*\theta$-continuous.*

Proof. Let $V \in O(Y)$.

$$\left.\begin{array}{r} V \in O(Y) \\ f \text{ is a.c.}e^*\theta\text{.c.} \end{array}\right\} \overset{\text{Theorem 3.1(7)}}{\Rightarrow}$$

$\Rightarrow f^{-1}[V] \subseteq f^{-1}[cl_\delta(V)] = e^*\text{-}int_\theta(e^*\text{-}int_\theta(f^{-1}[cl(V)])) \subseteq e^*\text{-}int_\theta(f^{-1}[V]) \subseteq f^{-1}[V]$

$\Rightarrow f^{-1}[V] = e^*\text{-}int_\theta(f^{-1}[V])$

$\Rightarrow f^{-1}[V] \in e^*\theta O(X)$. □

We recall that a topological space is said to be P_Σ [29] if for any open set V of X and each $x \in V$, there exists a regular closed set F of X containing x such that $x \in F \subseteq V$.

Theorem 3.5. *If $f : X \to Y$ is an almost contra $e^*\theta$-continuous function and Y is P_Σ, then f is $e^*\theta$-continuous.*

Proof. Let $V \in O(Y)$.

$$\left.\begin{array}{r} \left.\begin{array}{r} y \in V \in O(Y) \overset{Y \text{ is } P_\Sigma}{\Rightarrow} (\exists F \in RC(Y,y))(F \subseteq V) \\ \mathcal{A} := \{F | y \in V \Rightarrow (\exists F \in RC(Y,y))(F \subseteq V)\} \end{array}\right\} \Rightarrow \quad \cup\mathcal{A} = V \\ f \text{ is a.c.}e^*\theta\text{.c} \end{array}\right\} \Rightarrow$$

$\Rightarrow f^{-1}[V] = \bigcup_{F\in\mathcal{A}} f^{-1}[F] \in e^*\theta O(X)$. □

Definition 3.2. A function $f : X \to Y$ is said to be:

a) R-map [6] if $f^{-1}[A]$ is regular closed in X for every regular closed A of Y,

b) weakly e^*-irresolute [22] if $f^{-1}[A]$ is $e^*\theta$-open in X for every $e^*\theta$-open set A of Y,

c) pre-$e^*\theta$-closed if $f[A]$ is $e^*\theta$-closed in Y for every $e^*\theta$-closed A of X.

Theorem 3.6. *Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Then the following properties hold:*

(1) If f is almost contra $e^*\theta$-continuous and g is an R-map, then $g \circ f : X \to Z$ is almost contra $e^*\theta$-continuous,

(2) If f is almost $e^*\theta$-continuous and g is a contra R-map, then $g \circ f : X \to Z$ is

almost contra $e^*\theta$-continuous,

(3) If f is weakly e^*-irresolute and g is almost contra $e^*\theta$-continuous, then $g \circ f : X \to Z$ is almost contra $e^*\theta$-continuous.

Proof. Routine. □

Theorem 3.7. *If $f : X \to Y$ is a pre-$e^*\theta$-closed surjection and $g : Y \to Z$ is a function such that $g \circ f : X \to Z$ is almost contra $e^*\theta$-continuous, then g is almost contra $e^*\theta$-continuous.*

Proof. Let $V \in RO(Z)$.

$$\left.\begin{array}{r} \left.\begin{array}{r} V \in RO(Z) \\ g \circ f \text{ is a.c.} e^*\theta\text{.c.} \end{array}\right\} \Rightarrow (gof)^{-1}[V] = f^{-1}[g^{-1}[V]] \in e^*\theta C(X) \\ f \text{ is pre-}e^*\theta\text{-closed surjection} \end{array}\right\} \Rightarrow$$

$$\Rightarrow f[f^{-1}[g^{-1}[V]]] = g^{-1}[V] \in e^*\theta C(Y).$$ □

Theorem 3.8. *Let $\{X_\alpha | \alpha \in \Lambda\}$ be any family of topological spaces. If $f : X \to \Pi X_\alpha$ is an almost contra $e^*\theta$-continuous function, then $Pr_\alpha \circ f : X \to X_\alpha$ is almost contra $e^*\theta$-continuous for each $\alpha \in \Lambda$ where Pr_α is the projection of ΠX_α onto X_α.*

Proof. Let $\alpha \in \Lambda$ and $U_\alpha \in RO(X_\alpha)$.

$$\left.\begin{array}{r} \alpha \in \Lambda \Rightarrow Pr_\alpha \text{ is open and continuous} \Rightarrow Pr_\alpha \text{ is } R\text{-map} \\ U_\alpha \in RO(X_\alpha) \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow Pr_\alpha^{-1}[U_\alpha] \in RO(\Pi X_\alpha) \\ f \text{ is a.c.} e^*\theta\text{.c.} \end{array}\right\} \Rightarrow (Pr_\alpha \circ f\ )^{-1}[U_\alpha] = f^{-1}[Pr_\alpha^{-1}[U_\alpha]] \in e^*\theta C(X).$$ □

Definition 3.3. A function $f : X \to Y$ is called weakly $e^*\theta$-continuous (briefly w.$e^*\theta$.c.) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a $U \in e^*\theta O(X, x)$ such that $f[U] \subseteq cl(V)$.

Theorem 3.9. *Let $f : X \to Y$ be a function. Then the following properties hold:*

(1) If f is almost contra $e^*\theta$-continuous, then it is weakly $e^*\theta$-continuous,

(2) If f is weakly $e^{*}\theta$-continuous and Y is extremally disconnected, then f is almost contra $e^{*}\theta$-continuous.

Proof. (1) Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left.\begin{array}{r}(x \in X)(V \in O(Y, f(x))) \Rightarrow cl(V) \in RC(Y, f(x)) \\ f \text{ is a.c.}e^{*}\theta\text{.c.}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow f^{-1}[cl(V)] \in e^{*}\theta O(X, x) \\ U := f^{-1}[cl(V)]\end{array}\right\} \Rightarrow (U \in e^{*}\theta O(X, x))(f[U] \subseteq cl(V)).$$

(2) Let $V \in RC(Y)$ and $x \in f^{-1}[V]$.

$$\left.\begin{array}{r}(V \in RC(Y))(x \in f^{-1}[V]) \Rightarrow (V \in RC(Y, f(x)))(cl(V) = V) \\ Y \text{ is extremally disconnected}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow cl(V) \in RO(Y, f(x)) \\ f \text{ is w.}e^{*}\theta\text{.c.}\end{array}\right\} \Rightarrow (\exists U \in e^{*}\theta O(X, x))(f[U] \subseteq cl(V) = V)$$

$$\Rightarrow (\exists U \in e^{*}\theta O(X, x))(U \subseteq f^{-1}[V])$$

$$\Rightarrow f^{-1}[V] \in e^{*}\theta O(X).$$

□

4. Some Fundamental Properties

Definition 4.1. A topological space X is said to be:

a) $e^{*}\theta$-T_0 if for any distinct pair of points x and y in X, there is an $e^{*}\theta$-open set U in X containing x but not y or an $e^{*}\theta$-open set V in X containing y but not x,

b) $e^{*}\theta$-T_1 if for any distinct pair of points x and y in X, there is an $e^{*}\theta$-open set U in X containing x but not y and an $e^{*}\theta$-open set V in X containing y but not x,

c) $e^{*}\theta$-T_2 (resp. e^{*}-T_2 [13, 14]) if for every pair of distinct points x and y, there exist two $e^{*}\theta$-open (resp. e^{*}-open) sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem 4.1. *For a topological space X, the following properties are equivalent:*

(1) (X, τ) is $e^{*}\theta$-T_0;

(2) (X, τ) is $e^{*}\theta$-T_1;

(3) (X, τ) is $e^{*}\theta$-T_2;

(4) (X,τ) *is* e^**-*T_2*;*

(5) For every pair of distinct points $x,y\in X$*, there exist* $U\in e^*O(X,x)$ *and* $V\in e^*O(X,y)$ *such that* $e^*\text{-}cl(U)\cap e^*\text{-}cl(V)=\emptyset$*;*

(6) For every pair of distinct points $x,y\in X$*, there exist* $U\in e^*R(X,x)$ *and* $V\in e^*R(X,y)$ *such that* $U\cap V=\emptyset$*;*

(7) For every pair of distinct points $x,y\in X$*, there exist* $U\in e^*\theta O(X,x)$ *and* $V\in e^*\theta O(X,y)$ *such that* $e^*\text{-}cl_\theta(U)\cap e^*\text{-}cl_\theta(V)=\emptyset$*.*

Proof. $(3)\Rightarrow(2)$: Obvious.

$(2)\Rightarrow(1)$: Obvious.

$(1)\Rightarrow(3)$: Let $x,y\in X$ and $x\neq y$.

$$\left.\begin{array}{r}(x,y\in X)(x\neq y)\\ (1)\end{array}\right\}\Rightarrow(\exists W\in e^*\theta O(X,x))(y\notin W)$$

$$\left.\begin{array}{r}\overset{\text{Lemma 2.4}}{\Rightarrow}(\exists U\in e^*R(X,x))(U=e^*\text{-}cl_\theta(U)\subseteq W)\\ V:=\setminus U=\setminus e^*cl_\theta(U)\end{array}\right\}\Rightarrow$$

$$\Rightarrow(U\in e^*\theta O(X,x))(V\in e^*\theta O(X,y))(U\cap V=\emptyset).$$

$(3)\Rightarrow(4)$: The proof is obvious since $e^*\theta O(X)\subseteq e^*O(X)$.

$(4)\Rightarrow(5)$: Let $x,y\in X$ and $x\neq y$.

$$\left.\begin{array}{r}(x,y\in X)(x\neq y)\\ X\text{ is }e^*\text{-}T_2\end{array}\right\}\Rightarrow(\exists U\in e^*O(X,x))(\exists V\in e^*O(X,y))(U\cap V=\emptyset)$$

$$\Rightarrow(\exists U\in e^*O(X,x))(\exists V\in e^*O(X,y))(U\subseteq\setminus V)$$

$$\Rightarrow(\exists U\in e^*O(X,x))(\exists V\in e^*O(X,y))(e^*\text{-}cl(U)\subseteq\setminus V)$$

$$\Rightarrow(\exists U\in e^*O(X,x))(\exists V\in e^*O(X,y))(e^*\text{-}int(e^*\text{-}cl(U))=e^*\text{-}cl(U)\subseteq e^*\text{-}int(\setminus V)$$

$$\Rightarrow(\exists U\in e^*O(X,x))(\exists V\in e^*O(X,y))(e^*\text{-}cl(U)\subseteq e^*\text{-}int(\setminus V)=\setminus e^*\text{-}cl(V)$$

$$\Rightarrow(\exists U\in e^*O(X,x))(\exists V\in e^*O(X,y))(e^*\text{-}int(U)\cap e^*\text{-}cl(V)=\emptyset).$$

$(5)\Rightarrow(6)$: Let $x,y\in X$ and $x\neq y$.

$$\left.\begin{array}{r}(x,y\in X)(x\neq y)\\(5)\end{array}\right\}\Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow(\exists U_1\in e^*O(X,x))(\exists V_1\in e^*O(X,y))(e^*\text{-}cl(U_1)\cap e^*\text{-}cl(V_1)=\emptyset)\\(U_2:=e^*\text{-}cl(U_1))(V_2:=e^*\text{-}cl(V_1))\end{array}\right\}\Rightarrow$$

$$\Rightarrow(\exists U_2\in e^*R(X,x))(\exists V_2\in e^*R(X,y))(U_2\cap V_2=\emptyset).$$

(6) $\Rightarrow$ (7) : Let $x,y\in X$ and $x\neq y$.

$$\left.\begin{array}{r}(x,y\in X)(x\neq y)\\(6)\end{array}\right\}\Rightarrow(\exists U\in e^*R(X,x))(\exists V\in e^*R(X,y))(U\cap V=\emptyset)$$

$$\Rightarrow(\exists U\in e^*\theta O(X,x))(\exists V\in e^*\theta O(X,y))(e^*\text{-}cl_\theta(U)\cap e^*\text{-}cl_\theta(V)=\emptyset).$$

(7) $\Rightarrow$ (3) : Obvious. □

Definition 4.2. A topological space X is said to be:

a) weakly Hausdorff [27] (briefly weakly-T_2) if every point of X is an intersection of regular closed sets of X,

b) s-Urysohn [2] if for each pair of distinct points x and y in X, there exist $U\in SO(X,x)$ and $V\in SO(X,y)$ such that $cl(U)\cap cl(V)=\emptyset$.

Theorem 4.2. *For a function $f:X\to Y$, the following properties hold:*

(1) If f is an almost contra $e^*\theta$-continuous injection of a topological space X into a s-Urysohn space Y, then X is $e^*\theta$-T_2,

(2) If f is an almost contra $e^*\theta$-continuous injection of a topological space X into a weakly Hausdorff space Y, then X is $e^*\theta$-T_1.

Proof. (1) Let $x,y\in X$ and $x\neq y$.

$$\left.\begin{array}{r}(x,y\in X)(x\neq y)\\f\text{ is injective}\end{array}\right\}\Rightarrow\left.\begin{array}{c}f(x)\neq f(y)\\Y\text{ is }s\text{-Urysohn}\end{array}\right\}\Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow(\exists V_1\in SO(Y,f(x)))((\exists V_2\in SO(Y,f(y)))(cl(V_1)\cap cl(V_2)=\emptyset)\\f\text{ is a.c.}e^*\theta\text{.c.}\end{array}\right\}\overset{\text{Theorem 3.1(4)}}{\Rightarrow}$$

$\Rightarrow (\exists U_1 \in e^*\theta O(X,x))(\exists U_2 \in e^*\theta O(X,y))(f[U_1] \cap f[U_2] \subseteq cl(V_1) \cap cl(V_2) = \emptyset)$

$\Rightarrow (\exists U_1 \in e^*\theta O(X,x))(\exists U_2 \in e^*\theta O(X,y))(f[U_1 \cap U_2] = f[U_1] \cap f[U_2] = \emptyset)$

$\Rightarrow (\exists U_1 \in e^*\theta O(X,x))(\exists U_2 \in e^*\theta O(X,y))(U_1 \cap U_2 = \emptyset)$.

(2) Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}\left.\begin{array}{r}(x,y \in X)(x \neq y) \\ f \text{ is injective}\end{array}\right\} \Rightarrow \quad f(x) \neq f(y) \\ Y \text{ is weakly-}T_2\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists V_1 \in RC(Y,f(x)))(\exists V_2 \in RC(Y,f(y)))(f(x) \notin V_2)(f(y) \notin V_1) \\ f \text{ is a.c.}e^*\theta\text{.c.}\end{array}\right\} \overset{\text{Theorem 3.1(3)}}{\Rightarrow}$$

$\Rightarrow (\exists U_1 \in e^*\theta O(X,x))(\exists U_2 \in e^*\theta O(X,y))(f[U_1] \subseteq V_1)(f[U_2] \subseteq V_2)(f(x) \notin V_2)(f(y) \notin V_1)$

$\Rightarrow (\exists U_1 \in e^*\theta O(X,x))(\exists U_2 \in e^*\theta O(X,y))(x \notin U_2)(y \notin U_1)$. □

Remark 2. [15] The intersection of two $e^*\theta$-open sets is not necessarily $e^*\theta$-open as shown in the following example.

Example 4.1. [15] *Let $X = \{a,b,c,d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. Although the subsets $\{b,c,d\}$ and $\{a,c,d\}$ are $e^*\theta$-open in X, the set $\{c,d\}$ which is the intersection of these sets is not $e^*\theta$-open in X.*

Definition 4.3. A topological space X is called an $e^*\theta c$-space if the intersection of any two $e^*\theta$-open sets is an $e^*\theta$-open set.

Theorem 4.3. *If $f, g : X \to Y$ are almost contra $e^*\theta$-continuous functions, X is an $e^*\theta c$-space and Y is s-Urysohn, then $E = \{x \in X | f(x) = g(x)\}$ is $e^*\theta$-closed in X.*

Proof. Let $x \notin E$.

$$\left.\begin{array}{r}x \notin E \Rightarrow f(x) \neq g(x) \\ Y \text{ is } s\text{-Urysohn}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists V_1 \in SO(Y,f(x)))(\exists V_2 \in SO(Y,g(x)))(cl(V_1) \cap cl(V_2) = \emptyset) \\ f \text{ and } g \text{ are a.c.}e^*\theta\text{.c.}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists U_1 \in e^*\theta O(X,x))(\exists U_2 \in e^*\theta O(X,x))(f[U_1]\cap g[U_2] \subseteq cl(V_1)\cap cl(V_2)=\emptyset) \\ X \text{ is } e^*\theta c\text{-space}\end{array}\right\}\Rightarrow$$

$\Rightarrow (\exists U := U_1 \cap U_2 \in e^*\theta O(X,x))(f[U]\cap g[U] \subseteq f[U_1]\cap g[U_2] = \emptyset)$

$\Rightarrow (\exists U \in e^*\theta O(X,x))(U \cap E = \emptyset)$

$\Rightarrow x \notin e^*\text{-}cl_\theta(E)$. □

We say that the product space $X = X_1 \times \ldots \times X_n$ has Property $P_{e^*\theta}$ if A_i is an $e^*\theta$-open set in a topological space X_i for $i = 1, 2, \ldots n$, then $A_1 \times \ldots \times A_n$ is also $e^*\theta$-open in the product space $X = X_1 \times \ldots \times X_n$.

Theorem 4.4. *Let $f : X_1 \to Y$ and $g : X_2 \to Y$ be two functions, where*
(i) $X = X_1 \times X_2$ has the Property $P_{e^*\theta}$,
(ii) Y is a Urysohn space,
(iii) f and g are almost contra $e^*\theta$-continuous,
then $A = \{(x_1, x_2) | f(x_1) = g(x_2)\}$ is $e^*\theta$-closed in the product space $X = X_1 \times X_2$.

Proof. Let $(x_1, x_2) \notin A$.

$$\left.\begin{array}{r}(x_1,x_2)\notin A \Rightarrow f(x_1)\neq g(x_2) \\ Y \text{ is Urysohn}\end{array}\right\}\Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists V_1 \in O(Y,f(x_1)))(\exists V_2 \in O(Y,g(x_2)))(cl(V_1)\cap cl(V_2)=\emptyset)(cl(V_1),cl(V_2)\in RC(Y)) \\ f \text{ and } g \text{ are a.c.}e^*\theta\text{.c.}\end{array}\right\}\Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (f^{-1}[cl(V_1)] \in e^*\theta O(X_1,x_1))(g^{-1}[cl(V_2)] \in e^*\theta O(X_2,x_2)) \\ X = X_1\times X_2 \text{ has the Property } P_{e^*\theta}\end{array}\right\}\Rightarrow$$

$\Rightarrow ((x_1,x_2) \in f^{-1}[cl(V_1)] \times g^{-1}[cl(V_2)] \in e^*\theta O(X))(f^{-1}[cl(V_1)] \times g^{-1}[cl(V_2)] \subseteq \setminus A)$

$\Rightarrow \setminus A \in e^*\theta O(X_1 \times X_2)$

$\Rightarrow A \in e^*\theta C(X_1 \times X_2)$. □

Theorem 4.5. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function, given by $g(x) = (x, f(x))$ for every $x \in X$. If g is almost contra $e^*\theta$-continuous, then f is almost contra $e^*\theta$-continuous.*

Proof. Let $V \in RO(Y)$.

$$\left.\begin{array}{r} V \in RO(Y) \Rightarrow X \times V \in RO(X \times Y) \\ g \text{ is a.c.}e^*\theta\text{.c.} \end{array}\right\} \Rightarrow f^{-1}[V] = g^{-1}[X \times V] \in e^*\theta C(X). \qquad \square$$

We recall that for a function $f : X \to Y$, the subset $\{(x, f(x))|x \in X\}$ of $X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 4.4. A function $f : X \to Y$ has an $e^*\theta$-closed graph if for each $(x, y) \notin G(f)$, there exist $U \in e^*\theta O(X, x)$ and $V \in O(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.

Lemma 4.1. *The graph $G(f)$ of a function $f : X \to Y$ is $e^*\theta$-closed if and only if for each $(x, y) \notin G(f)$, there exist $U \in e^*\theta O(X, x)$ and $V \in O(Y, y)$ such that $f[U] \cap V = \emptyset$.*

Proof. Straightforward. $\square$

Theorem 4.6. *Let X and Y be two topological spaces. If $f : X \to Y$ is a function with an $e^*\theta$-closed graph, then $\{f(x)\} = \cap\{cl(f[U])|U \in e^*\theta O(X, x)\}$ for each x in X.*

Proof. Let $G(f)$ be $e^*\theta$-closed. Suppose that there exists a point of x in X such that $\{f(x)\} \neq \cap\{cl(f[U])|U \in e^*\theta O(X, x)\}$.

$$\{f(x)\} \neq \cap\{cl(f[U])|U \in e^*\theta O(X, x)\} \Rightarrow (\exists y \in \cap\{cl(f[U])|U \in e^*\theta O(X, x)\})(y \neq f(x))$$

$$\left.\begin{array}{r} \Rightarrow (\forall U \in e^*\theta O(X, x))(y \in cl(f[U]))((x, y) \notin G(f)) \\ G(f) \text{ is } e^*\theta\text{-closed} \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists V \in O(Y, y))(y \in cl(f[U]))(\emptyset = f[U] \cap V = cl(f[U]) \cap V \neq \emptyset)$$

This is a contradiction. $\square$

Theorem 4.7. *If $f : X \to Y$ is almost contra $e^*\theta$ -continuous and Y is Hausdorff, then $G(f)$ is $e^*\theta$-closed.*

Proof. Let $(x, y) \notin G(f)$.

$$\left.\begin{array}{r}(x,y)\notin G(f)\Rightarrow y\neq f(x)\\ Y \text{ is Hausdorff}\end{array}\right\}\Rightarrow(\exists U\in O(Y,y))(\exists V\in O(Y,f(x)))(U\cap V=\emptyset)$$

$$\Rightarrow (f(x)\notin Y\setminus cl(V))(U\subseteq Y\setminus cl(V)\in RO(Y))\Rightarrow f(x)\notin rker(U)$$

$$\left.\begin{array}{r}\Rightarrow x\notin f^{-1}[rker(U)]\overset{f\text{ is a.c.}e^*\theta\text{.c.}}{\Rightarrow} x\notin e^*\text{-}cl_\theta(f^{-1}[U])\\ V:=\backslash e^*\text{-}cl_\theta(f^{-1}[U])\end{array}\right\}\Rightarrow$$

$$\Rightarrow (V\in e^*\theta O(X,x))(U\in O(Y,y))(V\times U\subseteq\backslash G(f))$$

$$\Rightarrow (V\in e^*\theta O(X,x))(U\in O(Y,y))((V\times U)\cap G(f)=\emptyset).$$

□

Theorem 4.8. *If $f : X \to Y$ have an $e^*\theta$-closed graph and injective, then X is $e^*\theta$-T_1.*

Proof. Let $x_1, x_2 \in X$ and $x_1 \neq x_2$.

$$\left.\begin{array}{r}\left.\begin{array}{r}(x_1,x_2\in X)(x_1\neq x_2)\\ f\text{ is injective}\end{array}\right\}\Rightarrow f(x_1)\neq f(x_2)\Rightarrow(x_1,f(x_2))\in(X\times Y)\setminus G(f)\\ G(f)\text{ is }e^*\theta\text{-closed}\end{array}\right\}\Rightarrow$$

$$\Rightarrow (\exists U\in e^*\theta O(X,x_1))(\exists V\in O(Y,f(x_2)))(f[U]\cap V=\emptyset)$$

$$\Rightarrow (\exists U\in e^*\theta O(X,x_1))(\exists V\in O(Y,f(x_2)))(U\cap f^{-1}[V]=\emptyset)$$

$$\Rightarrow (\exists U\in e^*\theta O(X,x_1))(x_2\notin U)$$

Then X is $e^*\theta$-T_0. On the other hand, the notions of $e^*\theta$-T_0 and $e^*\theta$-T_1 are equivalent from Theorem 4.1. Thus X is $e^*\theta$-T_1. □

Theorem 4.9. *If $f : X \to Y$ has an $e^*\theta$-closed graph and X is an $e^*\theta c$-space, then $f^{-1}[K]$ is $e^*\theta$-closed for every compact subset K of Y.*

Proof. Let K be a compact subset of Y and let $x \notin f^{-1}[K]$.

$$\left.\begin{array}{r} x \notin f^{-1}[K] \Rightarrow f(x) \notin K \Rightarrow (\forall y \in K)(y \neq f(x)) \Rightarrow (x,y) \in (X \times Y) \setminus G(f) \\ G(f) \text{ is } e^*\theta\text{-closed} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\exists U_y \in e^*\theta O(X,x))(\exists V_y \in O(Y,y))(f[U_y] \cap V_y = \emptyset) \\ \mathcal{A} := \{V_y | y \in K\} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \left.\begin{array}{r} \Rightarrow (\mathcal{A} \subseteq O(Y))(K \subseteq \cup\mathcal{A}) \\ K \text{ is compact} \end{array}\right\} \Rightarrow (\exists \mathcal{A}^* \subseteq \mathcal{A})(|\mathcal{A}^*| < \aleph_0)(K \subseteq \cup\mathcal{A}^*) \\ U := \cap\{U_{y_i} | i = 1, 2, \ldots, n\} \end{array}\right\} \overset{X \text{ is } e^*\theta c\text{-space}}{\Rightarrow}$$

$$\Rightarrow (U \in e^*\theta O(X,x))(f[U] \cap K = \emptyset)$$

$$\Rightarrow (U \in e^*\theta O(X,x))(U \cap f^{-1}[K] = \emptyset)$$

$$\Rightarrow (U \in e^*\theta O(X,x))(U \subseteq \setminus f^{-1}[K])$$

$$\Rightarrow x \in e^*\text{-}int_\theta(X \setminus f^{-1}[K])$$

$$\overset{\text{Lemma 2.3(7)}}{\Rightarrow} x \in X \setminus e^*\text{-}cl_\theta(f^{-1}[K])$$

$$\Rightarrow x \notin e^*\text{-}cl_\theta(f^{-1}[K]).$$ □

Definition 4.5. A topological space X is said to be:

$a)$ strongly $e^*\theta C$-compact if every $e^*\theta$ -closed cover of X has a finite subcover (resp. $A \subseteq X$ is strongly $e^*\theta C$-compact if the subspace A is strongly $e^*\theta C$-compact),

$b)$ nearly compact [26] if every regular open cover of X has a finite subcover.

Theorem 4.10. *If $f : X \to Y$ is an almost contra $e^*\theta$-continuous surjection and X is strongly $e^*\theta C$-compact, then Y is nearly compact.*

Proof. Let $\mathcal{B} \subseteq RO(Y)$ and $Y = \cup\mathcal{B}$.

$$\left.\begin{array}{r} \left.\begin{array}{r} (\mathcal{B} \subseteq RO(Y))(Y = \cup\mathcal{B}) \\ f \text{ is a.c.}e^*\theta\text{.c.} \end{array}\right\} \Rightarrow (\mathcal{A} := \{f^{-1}[B] | B \in \mathcal{B}\} \subseteq e^*\theta C(X))(X = \cup\mathcal{A}) \\ X \text{ is strongly } e^*\theta C\text{-compact} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists \mathcal{A}^* \subseteq \mathcal{A})(|\mathcal{A}^*| < \aleph_0)(X = \cup \mathcal{A}^*) \\ f \text{ is surjective}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\mathcal{B}^* := \{f[A] | A \in \mathcal{A}^*\} \subseteq \mathcal{B})(|\mathcal{B}^*| < \aleph_0)(Y = \cup \mathcal{B}^*).$$ □

We recall that a topological space X is said to be almost regular [25] if for each regular closed set F of X and each point $x \in X \setminus F$, there exist disjoint open sets U and V such that $F \subseteq V$ and $x \in U$.

Theorem 4.11. *If a function $f : X \to Y$ is almost contra $e^*\theta$-continuous and Y is almost regular, then f is almost $e^*\theta$-continuous.*

Proof. Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left.\begin{array}{r}(x \in X)(V \in O(Y, f(x))) \\ Y \text{ is almost regular}\end{array}\right\} \overset{\text{Lemma 2.8}}{\Rightarrow}$$

$$\left.\begin{array}{r}\Rightarrow (\exists W \in RO(Y, f(x)))(cl(W) \subseteq int(cl(V))) \\ f \text{ is a.c.}e^*\theta\text{.c.}\end{array}\right\} \overset{\text{Theorem 3.1(3)}}{\Rightarrow}$$

$$\Rightarrow (\exists U \in e^*\theta O(X, x))(f[U] \subseteq cl(W) \subseteq int(cl(V))).$$ □

Definition 4.6. The $e^*\theta$-frontier of a subset A, denoted by $Fr_{e^*\theta}(A)$, is defined as $Fr_{e^*\theta}(A) = e^*\text{-}cl_\theta(A) \setminus e^*\text{-}int_\theta(A)$, equivalently $Fr_{e^*\theta}(A) = e^*\text{-}cl_\theta(A) \cap e^*\text{-}cl_\theta(X \setminus A)$.

Theorem 4.12. *The set of points $x \in X$ on which $f : X \to Y$ is not almost contra $e^*\theta$-continuous is identical with the union of the $e^*\theta$-frontiers of the inverse images of regular closed sets of Y containing $f(x)$.*

Proof. Let $A := \{x | f \text{ is not a.c.}e^*\theta\text{.c. at } x \in X\}$.

$$\begin{array}{rcl} x \in A & \Rightarrow & f \text{ is not a.c.}e^*\theta\text{.c. at } x \\ & \Rightarrow & (\exists V \in RC(Y, f(x)))(\forall U \in e^*\theta O(X, x))(f[U] \nsubseteq V) \\ & \Rightarrow & (\exists V \in RC(Y, f(x)))(\forall U \in e^*\theta O(X, x))(U \cap (X \setminus f^{-1}[V]) \neq \emptyset) \\ & \Rightarrow & (x \in f^{-1}[V])(x \in e^*\text{-}cl_\theta(X \setminus f^{-1}[V]) = X \setminus e^*\text{-}int_\theta(f^{-1}[V])) \\ & \Rightarrow & x \in Fr_{e^*\theta}(f^{-1}[V]) \end{array}$$

Then we have $A \subseteq \cup \{Fr_{e^*\theta}(f^{-1}[V]) | V \in RC(Y, f(x))\} \dots (*)$

$$\left.\begin{array}{r} x \notin A \Rightarrow f \text{ is a.c.}e^*\theta\text{.c. at } x \\ V \in RC(Y, f(x)) \end{array}\right\} \Rightarrow (\exists U \in e^*\theta O(X, x))(U \subseteq f^{-1}[V])$$

$\Rightarrow x \in e^*\text{-}int_\theta(f^{-1}[V])$

$\Rightarrow x \notin Fr_{e^*\theta}(f^{-1}[V])$

$\Rightarrow x \notin \cup \{Fr_{e^*\theta}(f^{-1}[V]) | V \in RC(Y, f(x))\}$

Then we have $\cup \{Fr_{e^*\theta}(f^{-1}[V]) | V \in RC(Y, f(x))\} \subseteq A \dots (**)$

$(*), (**) \Rightarrow A = \cup \{Fr_{e^*\theta}(f^{-1}[V]) | V \in RC(Y, f(x))\}$. □

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