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ARTICLE

Numerical approach for solving linear Fredholm integro-differential equation with piecewise intervals by Bernoulli polynomials

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ABSTRACT

In this paper a numerical method is given for the solution of linear Fredholm integro-differential equation (FIDE) with piecewise intervals under the mixed conditions using the Bernoulli polynomials. The aim of this article is to present an efficient numerical procedure for solving linear FIDE with piecewise intervals. This method transforms linear FIDE with piecewise intervals and the given conditions into matrix equation which corresponds to a system of linear algebraic equation. Finally, some experiments and their numerical solutions are given. The results reveal that this method is reliable and efficient.

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1. Introduction

In recent years, there has been a growing interest in the Fredholm integro-differential equations (FIDEs). FIDEs are an equation that the unknown function appears under the sign of integration and it also contains the derivatives functional arguments of the unknown function.

Many physical problems are modelled by integral or integro differential equations. Historically, they have achieved great popularity among mathematicians and physicists in formulating boundary value problems of gravitation, electrostatics, fluid dynamics, scattering, engineering, biology, medicine [1,6,7,14,17–20,22,24,25], economics, potential theory and many others [9,12,13,15,26]. The some initial-value and boundary value problems can be transformed into a Fredholm integral equations or FIDEs. Moreover, These type equations usually difficult to solve analytically, so we need a reliable numerical method. By the given reasons, many scientists have been motivated that they have been studied many numerical methods to solve FIDEs. Every methods have advantages or disadvantages but this did not stop scientists on the contrary it has led to the development of various methods such as Wavelet-Galerkin method [5], Tau method [11], Spline method [8], reproducing kernel algorithm [2,4], collocation methods of different polynomials such as Chebyshev [3], Laguerre [10], Taylor [21].

The technique that we used is the numerical solution method, which is based on numerical solution of linear Fredholm differential equations with variable coefficients in terms of Bernoulli polynomials.

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In this study, the basic ideas of the above studies are developed and applied to the mth-order linear FIDE with piecewise intervals with variable coefficients

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = g(x) + \sum_{j=0}^{J} \lambda_j \int_{a_j}^{b_j} K_j(x,t)y(t)\,\mathrm{d}t, \tag{1}$$

where $a \le x, t \le b$ and $a \le a_j < b_j \le b$, under the mixed conditions, for $i = 0, 1, 2, \dots, m-1$

$$\sum_{k=0}^{m-1} [a_{ik}y^{(k)}(a) + b_{ik}y^{(k)}(b)] = \mu_i \tag{2}$$

and the solution is expressed in the form

$$y(x) = \sum_{n=0}^{N} a_n B_n(x), \tag{3}$$

which is a Bernoulli polynomial of degree N and a_n are unknown Bernoulli coefficients.

2. Bernoulli polynomials

The Bernoulli polynomials are defined by the generating function [16,23]

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n, \quad |t| < 2\pi$$

or equivalently

$$B_N(x) = \sum_{i=0}^{N} \binom{N}{i} b_N x^{N-i},$$

where b_N are Bernoulli numbers.

The first a few Bernoulli polynomials are

$$B_0(x) = 1, \quad B_1(x) = x - \tfrac{1}{2}, \quad B_2(x) = x^2 - x + \tfrac{1}{6}, \quad B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x$$
$$B_4(x) = x^4 - 2x^3 + x^2 - \tfrac{1}{30}, \quad B_5(x) = x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x$$

using these Bernoulli polynomials, we can find the Bernoulli numbers which is define $b_N = B_N = B_N(0)$. For example

$$b_0 = B_0 = B_0(0) = 1, \quad b_1 = B_1 = B_1(0) = -\tfrac{1}{2}, \quad b_2 = B_2 = B_2(0) = \tfrac{1}{6}, \quad b_3 = B_3 = B_3(0) = 0,$$
$$b_4 = B_4 = B_4(0) = -\tfrac{1}{30}, \quad b_5 = B_5 = B_5(0) = 0.$$

3. Fundamental matrix relations

Let us consider the mth-order FIDE with variable coefficients (1) and find the matrix forms of each term of equation

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = g(x) + \sum_{j=0}^{J} \lambda_j \int_{a_j}^{b_j} K_j(x,t)y(t)\,\mathrm{d}t, \quad a \le x, t \le b, a \le a_j < b_j \le b$$

or shortly

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = g(x) + \sum_{j=0}^{J} \lambda_j I_j(x), \tag{4}$$

where

$$I_j(x) = \int_{a_j}^{b_j} K_j(x,t)y(t)\,\mathrm{d}t. \tag{5}$$

And then we write matrix form of $y(x)$

$$y(x) = \mathbf{B}(x)\mathbf{A}, \tag{6}$$

where

$$\mathbf{B}(x) = [B_0(x) \quad B_1(x) \quad \cdots \quad B_N(x)], \tag{7}$$

$$\mathbf{A} = [a_0 \quad a_1 \quad \cdots \quad a_N]^{\mathrm{T}}. \tag{8}$$

By using the general representation of Bernoulli polynomials which is defined by

$$B_N(x) = \sum_{i=0}^{N} \binom{N}{i} b_N x^{N-i}.$$

In Equation (6) we can write for variable t

$$y(t) = \mathbf{B}(t)\mathbf{A} \tag{9}$$

and using the Maclaurin expansion we obtain

$$K_j(x,t) = \mathbf{X}(x)\mathbf{K}_t\mathbf{X}^{\mathrm{T}}(t), \quad \mathbf{K}_t = [k_{ij}^t], \quad i,j = 0,1,2,\ldots,N \tag{10}$$

and then using the Bernoulli expansion

$$K_j(x,t) = \mathbf{B}(x)\mathbf{K}_f\mathbf{B}^{\mathrm{T}}(t), \quad \mathbf{K}_f = [k_{ij}^f], \quad i,j = 0,1,2,\ldots,N \tag{11}$$

and then

$$\mathbf{X}(x)\mathbf{K}_t\mathbf{X}^{\mathrm{T}}(t) = \mathbf{B}(x)\mathbf{K}_f\mathbf{B}^{\mathrm{T}}(t)$$
$$\mathbf{X}(x)\mathbf{K}_t\mathbf{X}^{\mathrm{T}}(t) = \mathbf{X}(x)\mathbf{S}\mathbf{K}_f\mathbf{S}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}(t)$$
$$\mathbf{K}_t = \mathbf{S}\mathbf{K}_f\mathbf{S}^{\mathrm{T}}$$
$$\mathbf{K}_f = \mathbf{S}^{-1}\mathbf{K}_t(\mathbf{S}^{\mathrm{T}})^{-1},$$

where

$$\mathbf{B}(x) = \mathbf{X}(x)\mathbf{S}$$
$$\mathbf{X}(x) = [1 \quad x \quad x^2 \quad x^3 \quad \cdots \quad x^N]$$
$$\mathbf{S} = \begin{bmatrix} \binom{0}{0}b_0 & \binom{1}{1}b_1 & \binom{2}{2}b_2 & \binom{3}{3}b_3 & \cdots & \binom{N}{N}b_N \\ 0 & \binom{1}{0}b_1 & \binom{2}{1}b_2 & \binom{3}{2}b_3 & \cdots & \binom{N}{N-1}b_N \\ 0 & 0 & \binom{2}{0}b_2 & \binom{3}{1}b_3 & \cdots & \binom{N}{N-2}b_N \\ 0 & 0 & 0 & \binom{3}{0}b_3 & \cdots & \binom{N}{N-3}b_N \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{N}{0}b_N \end{bmatrix}.$$

Moreover, using this relation we can find

$$\mathbf{X}^{(k)}(x) = \mathbf{X}(x)\mathbf{M}^k,$$
$$y^{(k)}(x) = \mathbf{X}(x)\mathbf{M}^k\mathbf{S}\mathbf{A}, \tag{12}$$

where

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By substituting Equation (5) into Equations (9) and (11), we obtain

$$I_j = \int_{a_j}^{b_j} \mathbf{B}(x)\mathbf{K}_f\mathbf{B}^{\mathrm{T}}(t)\mathbf{A}\,\mathrm{d}t = \mathbf{B}(x)\mathbf{K}_f\mathbf{Q}_f\mathbf{A}, \tag{13}$$

where

$$\mathbf{Q}_j = \int_{a_j}^{b_j} \mathbf{B}^{\mathrm{T}}(t)\mathbf{B}(t)\,\mathrm{d}t. \tag{14}$$

We can write Equation (14) also

$$\mathbf{Q}_j = \int_{a_j}^{b_j} \mathbf{B}^{\mathrm{T}}(t)\mathbf{B}(t)\,\mathrm{d}t = \int_{a_j}^{b_j} \mathbf{S}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}(t)\mathbf{X}(t)\mathbf{S}\,\mathrm{d}t = \mathbf{S}^{\mathrm{T}}\underbrace{\left[\int_{a_j}^{b_j} \mathbf{X}^{\mathrm{T}}(t)\mathbf{X}(t)\,\mathrm{d}t\right]}_{\mathbf{H}_j}\mathbf{S}, \tag{15}$$

where

$$\mathbf{H}_j = [h_{kl}^j(x)], \quad h_{kl}^j(x) = \int_{a_j}^{b_j} \mathbf{X}^{\mathrm{T}}(t)\mathbf{X}(t)\,\mathrm{d}t = \frac{b_j^{k+l+1} - a_j^{k+l+1}}{k+l+1}, \quad k,l = 0,1,2,\ldots,N. \tag{16}$$

By substituting Equation (15) into Equation (16) we obtain

$$\mathbf{Q}_j = \mathbf{S}^{\mathrm{T}}\mathbf{H}_j\mathbf{S}.$$

It is known that by Equation (11)

$$\mathbf{B}(x) = \mathbf{X}(x)\mathbf{S}$$

and using this relation we find

$$\mathbf{I}_j(x) = \mathbf{X}(x)\mathbf{S}\mathbf{K}_f\mathbf{Q}_j\mathbf{A} \tag{17}$$

so that, the matrix representation of

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = g(x) + \sum_{j=0}^{J} \lambda_j \int_{a_j}^{b_j} K_j(x,t)y(t)\,\mathrm{d}t$$

can be given by

$$\sum_{k=0}^{m} \mathbf{P}_k y^{(k)} = \mathbf{G} + \sum_{j=0}^{J} \lambda_j \mathbf{I}_j. \tag{18}$$

3.1. Matrix representation of the conditions

Using the relation (12), the matrix form of the conditions given by Equation (2) can be written as

$$\sum_{k=0}^{m-1} [a_{ik}\mathbf{X}(a) + b_{ik}\mathbf{X}(b)]\mathbf{M}^k\mathbf{S}\mathbf{A} = \mu_i, \quad i = 0, 1, 2, \dots, m-1. \tag{19}$$

4. Method of solution

We are ready to construct the fundamental matrix equation corresponding to Equation (1). For this purpose, firstly we write

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = g(x) + \sum_{j=0}^{J} \lambda_j I_j(x)$$

and then we can write

$$x = x_i = a + \frac{b-a}{N}i, \quad i = 0, 1, 2, \dots, N$$

$$\sum_{k=0}^{m} P_k(x_i)y^{(k)}(x_i) = g(x_i) + \sum_{j=0}^{J} \lambda_j I_j(x_i), \quad i = 0, 1, 2, \dots, N$$

and then the fundamental matrix equation is gained by

$$\sum_{k=0}^{m} \mathbf{P}_k y^{(k)} = \mathbf{G} + \sum_{j=0}^{J} \lambda_j \mathbf{I}_j, \tag{20}$$

where

$$\mathbf{P}_k = \begin{bmatrix} P_k(x_0) & 0 & 0 & \cdots & 0 \\ 0 & P_k(x_1) & 0 & \cdots & 0 \\ 0 & 0 & P_k(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & P_k(x_N) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ g(x_2) \\ \vdots \\ g(x_N) \end{bmatrix}.$$

By substituting Equation (20) into Equations (12) and (17), we obtain

$$\sum_{k=0}^{m} \mathbf{P}_k \mathbf{X} \mathbf{M}^k \mathbf{S} \mathbf{A} = \mathbf{G} + \sum_{j=0}^{J} \lambda_j (\mathbf{X} \mathbf{S} \mathbf{K}_j \mathbf{Q}_j \mathbf{A}),$$

$$\underbrace{\left(\sum_{k=0}^{m} \mathbf{P}_k \mathbf{X} \mathbf{M}^k \mathbf{S} - \sum_{j=0}^{J} \lambda_j \mathbf{X} \mathbf{S} \mathbf{K}_j \mathbf{Q}_j \right)}_{\mathbf{W}_f} \mathbf{A} = \mathbf{G}. \tag{21}$$

The fundamental matrix equation (21) for Equation (1) corresponds to a system of $(N+1)$ algebraic equation for the $(N+1)$ unknown coefficients . Briefly, we can write Equation (21)

$$\mathbf{W}_f \mathbf{A} = \mathbf{G} \quad \text{or} \quad [\mathbf{W}_f; \mathbf{G}] \tag{22}$$

and the matrix form for conditions (2) is,

$$\mathbf{U}_i \mathbf{A} = [\mu_i] \quad \text{or} \quad [\mathbf{U}_i; \mu_i], \quad i = 0, 1, 2, \ldots, m-1, \tag{23}$$

where

$$\mathbf{U}_i = \sum_{k=0}^{m} [a_{ik} \mathbf{X}(a) + b_{ik} \mathbf{X}(b)] \mathbf{M}^k \mathbf{S} = [u_{i0} \quad u_{i1} \quad \cdots \quad u_{iN}].$$

To obtain the solution of Equation (1) under the conditions (2), by replacing the rows matrices (23) by the last m rows of the matrix (22), we have the required augmented matrix

$$[\mathbf{W}_f^*; \mathbf{G}^*] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & \cdots & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & w_{12} & \cdots & w_{1N} & ; & g(x_1) \\ \vdots & \vdots & \vdots & \cdots & \vdots & ; & \vdots \\ w_{(N-m)0} & w_{(N-m)1} & w_{(N-m)2} & \cdots & w_{(N-m)N} & ; & g(x_{N-m}) \\ u_{00} & u_{01} & u_{02} & \cdots & u_{0N} & ; & \mu_0 \\ u_{10} & u_{11} & u_{12} & \cdots & u_{1N} & ; & \mu_1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & ; & \vdots \\ u_{(m-1)0} & u_{(m-1)1} & u_{(m-1)2} & \cdots & u_{(m-1)N} & ; & \mu_{m-1} \end{bmatrix}$$

or corresponding matrix equation

$$\mathbf{W}_f^* \mathbf{A} = \mathbf{G}^*. \tag{24}$$

If $\operatorname{rank}(\mathbf{W}_f^*) = \operatorname{rank}[\mathbf{W}_f^*; \mathbf{G}^*] = N+1$, then we can write

$$\mathbf{A} = (\mathbf{W}_f^*)^{-1} \mathbf{G}^*. \tag{25}$$

Thus the coefficients a_i, $i = 0, 1, 2, \ldots, N$ are uniquely determined by Equation (25). Also Equation (1) with conditions Equation (2) has a unique solution. This solution is given by truncated Bernoulli series equation (3). We can easily check the accuracy of the suggested method. Since the truncated Bernoulli series (3) is an approximate solution of Equation (1), when the solution $y_N(x)$ and its derivatives are substituted in Equation (1) the resulting equation must be satisfied approximately, that is for $x = x_q \in [0,1]$, $q = 0, 1, 2, \ldots$

$$E(x_q) = \left| \sum_{k=0}^{m} P_k(x) y^{(k)}(x) - g(x) - \sum_{j=0}^{J} \lambda_j \int_{a_j}^{b_j} K_j(x,t) y(t)\, \mathrm{d}t \right| \cong 0 \tag{26}$$

and $E(x_q) \le 10^{-k_q}$ (k_q positive integer). If $\max 10^{-k_q} = 10^{-k}$(k positive integer) is prescribed, then the truncation limit N is increased until the difference $E(x_q)$ at each of the points becomes smaller than the prescribed 10^{-k}. On the other hand, the error can be estimated by the function

$$E_N(x) = \sum_{k=0}^{m} P_k(x) y^{(k)}(x) - g(x) - \sum_{j=0}^{J} \lambda_j \int_{a_j}^{b_j} K_j(x,t) y(t)\, \mathrm{d}t \tag{27}$$

if $E_N(x) \to 0$ when N is sufficiently large enough then the error decreases.

5. Illustrative examples

In this section, several numerical examples are given to illustrate the accuracy and effectiveness properties of method and all of them performed on the computer using a program written in Maple17. The absolute errors in tables are the values of $|y(x) - y_N(x)|$ at selected points.

Example 5.1: Let us consider the linear FIDE with piecewise intervals,

$$y''(x) + xy'(x) + y + 6\int_{-1}^{0} xty(t)\, \mathrm{d}t - 6\int_{0}^{1} y(t)\, \mathrm{d}t = 6x^2 + x - 3 \tag{28}$$

with initial conditions,

$$y(0) = 0, \quad y'(0) = 1. \tag{29}$$

Then, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = 1$, $g(x) = 6x^2 + 4x - 3$, $K_1(x,t) = xt$, $K_2(x,t) = 1$, $\lambda_1 = -6$ and $\lambda_2 = 6$ for $N = 3$ collocation points are $\{x_0 = 0, \quad x_1 = \frac{1}{3}, \quad x_2 = \frac{2}{3}, \quad x_3 = 1\}$ and the fundamental matrix equation of the problem is,

$$\underbrace{\left(\sum_{k=0}^{m} \mathbf{P}_k \mathbf{X} \mathbf{M}^k \mathbf{S} - \sum_{j=0}^{J} \lambda_j \mathbf{X} \mathbf{S} \mathbf{K}_j \mathbf{Q}_j \right)}_{\mathbf{W}_f} \mathbf{A} = \mathbf{G}, \tag{30}$$

where,

$$\mathbf{P}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{P}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 1/9 & 1/27 \\ 1 & 2/3 & 4/9 & 8/27 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 1 & -1/2 & 1/6 & 0 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 1 & -3/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{K}_1 = \begin{bmatrix} 1/4 & 1/2 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} -3 \\ -2 \\ 1/3 \\ 4 \end{bmatrix}$$

$$\mathbf{Q}_1 = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 13/12 & -7/6 & 149/120 \\ 1 & -7/6 & 241/180 & -3/2 \\ -1 & 149/120 & -3/2 & 1471/840 \end{bmatrix}, \quad \mathbf{Q}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/12 & 0 & -1/120 \\ 0 & 0 & 1/180 & 0 \\ 0 & -1/120 & 0 & 1/840 \end{bmatrix}$$

and conditions matrices are

$$\begin{bmatrix} \mathbf{U}_0 & ; & \lambda_0 \\ \mathbf{U}_1 & ; & \lambda_1 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & 1/6 & 0 & ; & 0 \\ 0 & 1 & -1 & 1/2 & ; & 1 \end{bmatrix}.$$

If these matrix are substituted in Equation (30), it is obtained linear algebraic system. We obtained the approximate solution of the problem for $N = 3$

$$\begin{aligned} y(x) &= a_0B_0(x) + a_1B_1(x) + a_2B_2(x) + a_3B_3(x) \\ y(x) &= (-0.3 \cdot 10^{-14}) + (0.99999999999999)x + (2.00000000000019)x^2 \\ &\quad + (0.666666666666732 \cdot 10^{-14})x^3. \end{aligned}$$

The exact solution of this problem is $y(x) = 2x^2 + x$ (see Table 1). Figure 1 shows the comparison between the exact solution and different for the N Bernoulli collocation method solutions of the system in Equation (30). It seems that solutions almost identical. One can obtain a better approximation to the numerical solutions by adding new terms to the series in Equation (3). On the other hand, Figure 1 shows that the comparison between the errors functions for various N. It seems that accuracy increases as the N increased. From Table 1, the results we obtained have shown speedy convergence. It is of interest to note that the method discussed above works effectively for linear models.

Example 5.2: Consider the second-order linear FIDE with piecewise intervals,

$$x^2y''(x) + xy'(x) + y(x) + \int_{-1/2}^{1/2} xty(t)\,dt - \int_{1/2}^{1} y(t)\,dt = g(x), \tag{31}$$

where $g(x) = e^x + xe^x + x^2e^x + 0.085435354218885x - 1.06956055775892$ with conditions

$$y(0) = 1, \quad y'(0) = 1 \tag{32}$$

Table 1. Numerical and error results of Example 5.1 for different N.

x	Exact solution	$N = 3$	$N_e = 3$	$N = 5$	$N_e = 5$	$N = 8$	$N_e = 8$
0.0	0.000000	0.000000	0.30E−14	0.000000	0.11E−14	0.000000	0.23E−15
0.1	0.120000	0.119999	0.20E−14	0.119999	0.30E−14	0.119999	0.40E−14
0.2	0.280000	0.280000	0.30E−14	0.279999	0.80E−14	0.279999	0.16E−13
0.3	0.480000	0.480000	0.11E−13	0.479999	0.18E−13	0.479999	0.36E−13
0.4	0.720000	0.720000	0.23E−13	0.719999	0.31E−13	0.719999	0.65E−13
0.5	1.000000	1.000000	0.40E−13	0.999999	0.47E−13	0.999999	0.10E−12
0.6	1.320000	1.320000	0.60E−13	1.319999	0.70E−13	1.319999	0.14E−12
0.7	1.680000	1.680000	0.80E−13	1.679999	0.90E−13	1.679999	0.19E−12
0.8	2.080000	2.080000	0.11E−12	2.079999	0.12E−12	2.079999	0.25E−12
0.9	2.520000	2.520000	0.14E−12	2.519999	0.13E−12	2.519999	0.31E−12
1.0	3.000000	3.000000	0.19E−12	2.999999	0.17E−12	2.999999	0.37E−12

and the exact solution is $y(x) = e^x$. Figure 2 shows that the comparison between the errors functions for various N. It seems that accuracy increases as the N increased. From Table 2, the results we obtained have shown speedy convergence. It is of interest to note that the method discussed above works effectively for linear models.

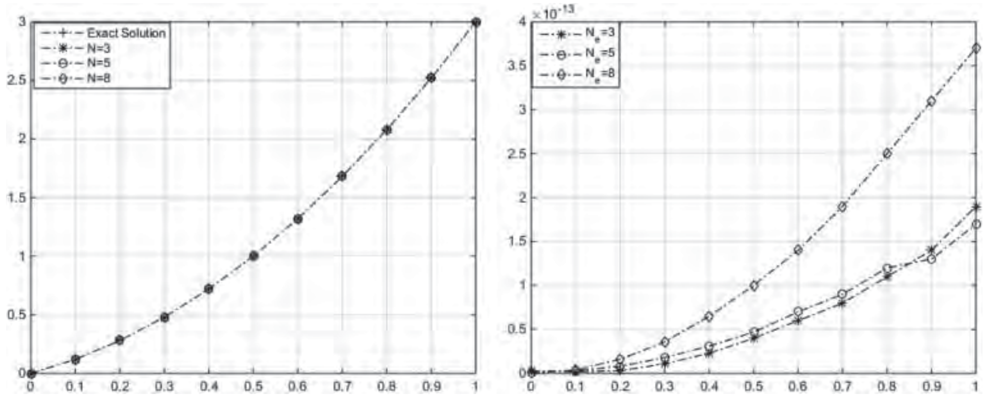

Figure 1. Numerical results of Example 5.1 for various N.

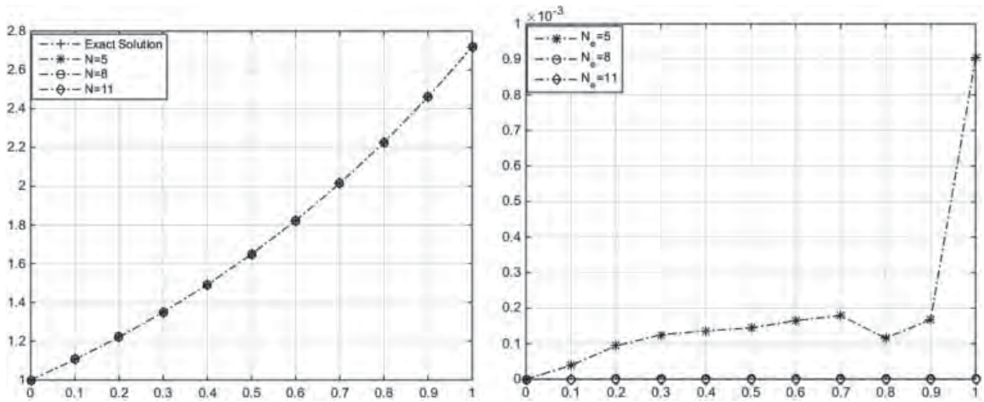

Figure 2. Numerical results of Example 5.2 for various N.

Table 2. Numerical and error results of Example 5.2 for different N.

x	Exact solution	$N = 5$	$N_e = 5$	$N = 8$	$N_e = 8$	$N = 11$	$N_e = 11$
0.0	1.000000	1.000000	0.000000	1.000000	0.80E−14	1.000000	0.000000
0.1	1.105171	1.105132	0.385E−4	1.105171	0.30E−11	1.105171	0.10E−14
0.2	1.221403	1.221309	0.939E−4	1.221403	0.229E−9	1.221403	0.20E−13
0.3	1.349859	1.349733	0.125E−3	1.349859	0.590E−9	1.349859	0.30E−13
0.4	1.491825	1.491689	0.136E−3	1.491825	0.940E−9	1.491825	0.40E−13
0.5	1.648721	1.648577	0.144E−3	1.648721	0.130E−8	1.648721	0.50E−13
0.6	1.822119	1.821954	0.165E−3	1.822119	0.164E−8	1.822119	0.60E−13
0.7	2.013753	2.013574	0.179E−3	2.013753	0.197E−8	2.013753	0.70E−13
0.8	2.225541	2.225426	0.115E−3	2.225541	0.236E−8	2.225541	0.70E−13
0.9	2.459603	2.459771	0.168E−3	2.459603	0.324E−9	2.459603	0.60E−13
1.0	2.718282	2.719186	0.905E−3	2.718282	0.215E−7	2.718282	0.10E−11

6. Steps of solutions

In this section Algorithms 6.1 and 6.2 have been given for calculations of Examples 5.1 and 5.2, respectively. Also, the algorithms can be applied any computer program.

Algorithm 6.1:
Step 1

(a) Input the number of truncated Bernoulli polynomial limit N ($N \in \mathrm{N}$)
(b) Determine the a, b, $P_0(x), P_1(x), \ldots, P_m(x)$, $g(x)$, $K_j(x,t)$, λ_j and the mixed conditions.
(c) The mixed conditions put in Equation (2)

Step 2

(a) Set the collocation points x_i, $i = 0, 1, \ldots, N$. There are $x_0 = a$ and $x_N = b$.

Step 3

(a) Construct the matrices $\mathbf{P}_k$, $\mathbf{X}$, $\mathbf{M}^k$, $\mathbf{S}$, $\mathbf{K}_j$, $\mathbf{Q}_j$. Equations (6)–(18)
(b) Compute $\mathbf{W}_f$ and $\mathbf{G}$ matrices.
(c) Construct the conditional m rows matrices equation (19)

Step 4

(a) Construct augmented matrix $[\mathbf{W}_f^*; \mathbf{G}^*]$ from Equation (24)

Step 5

(a) If rank $(\mathbf{W}_f^*) = \text{rank}\ [\mathbf{W}_f^*; \mathbf{G}^*] = N + 1$ then, to solve the (or solve the system by using Gauss elimination method).

Step 6

(a) Substituting all elements of the Bernoulli coefficients matrix solution as, respectively, into Equation (3). Finally, this will be our solution.

Algorithm 6.2:
Step 1

(a) Input the number of truncated Bernoulli polynomial limit N ($N \in \mathrm{N}$).
(b) Determine the a, b, $P_0(x), P_1(x), \ldots, P_m(x)$, $g(x)$, $K_j(x,t)$,λ_j and the mixed conditions.
(c) The mixed conditions put in Equation (2)

Step 2

(a) Set the collocation points x_i, $i = 0, 1, \ldots, N$. There are $x_0 = a$ and $x_N = b$.

Step 3

(a) Construct the matrices $\mathbf{P}_k$, $\mathbf{X}$, $\mathbf{M}^k$, $\mathbf{S}$, $\mathbf{K}_j$, $\mathbf{Q}_j$. Equations (6)–(18)
(b) Compute $\mathbf{W}_f$ and $\mathbf{G}$ matrices

(c) Construct the conditional m rows matrices equation (19)

Step 4

(a) Construct augmented matrix $[\mathbf{W}_f^*; \mathbf{G}^*]$ from Equation (24)

Step 5

(a) If rank $(\mathbf{W}_f^*) = \text{rank }[\mathbf{W}_f^*; \mathbf{G}^*] = N + 1$ then, to solve the (or solve the system by using Gauss elimination method).

Step 6

(a) Substituting all elements of the Bernoulli coefficients matrix solution as, respectively, into Equation (3). Finally, this will be our solution.

7. Conclusion

In recent years, the studies of high-order linear FIDE with piecewise intervals have attracted the attention of many mathematicians and physicists. The Bernoulli collocation method has been presented to solve linear FIDE with piecewise intervals. One of the advantages of this method is that numerical solution of the integro-differential equations can be converted into system of linear algebraic equations. Another considerable advantage of this method is to obtain the analytical solutions if the equation has an exact solution that is a polynomial function as in Example 1. Application of the given method allows the creation of more effective and faster algorithms than the ordinary ones. Moreover, another considerable advantage of this method is the Bernoulli polynomial coefficients of the solution are found very easily, shorter computation times are so low such as 1.1 sn for Example 2 (CPU Core2 Duo 2.13 Ghz, RAM 2 Gb) and lower operation count results in reduction of cumulative truncation errors and improvement of overall accuracy.

Illustrative examples are included to demonstrate validity and applicability of the technique and performed on the computer using a program written in Maple 17. The method can also extended to the system of linear FIDEs with variable coefficients, but some modifications are required.

Disclosure statement

No potential conflict of interest was reported by the authors.

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