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New estimators based on order statistics in some families of scale distributions

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ABSTRACT
In this study some new unbiased estimators based on order statistics are proposed for the scale parameter in some family of scale distributions. These new estimators are suitable for the cases of complete (uncensored) and symmetric doubly Type-II censored samples. Further, they can be adapted to Type II right or Type II left censored samples. In addition, unbiased standard deviation estimators of the proposed estimators are also given. Moreover, unlike BLU estimators based on order statistics, expectation and variance-covariance of relevant order statistics are not required in computing these new estimators.
Simulation studies are conducted to compare performances of the new estimators with their counterpart BLU estimators for small sample sizes. The simulation results show that most of the proposed estimators in general perform almost as good as the counterpart BLU estimators; even some of them are better than BLU in some cases. Furthermore, a real data set is used to illustrate the new estimators and the results obtained parallel with those of BLUE methods.

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1. Introduction

In this paper estimation of scale parameter σ for scale family of distributions is concerned. A scale family is characterized with a distribution function F and its associated probability density function f in the family that satisfy

$$F(x;\sigma) = F_0\left(\frac{x}{\sigma}\right) \text{ and } f(x;\sigma) = \frac{1}{\sigma} f_0\left(\frac{x}{\sigma}\right), \tag{1}$$

where F_0 and f_0 are, respectively, the distribution and probability density functions that do not depend on the scale parameter σ. The parameter σ need not be standard deviation. In the presence of censored data, which is frequently the case in the area of reliability and life testing, one alternative and useful method for estimation in family of scale distributions is to use BLUE (Best Linear Unbiased Estimation) method based on order statistics.

BLUE method, however, needs expected values, variance-covariance matrix of order statistics and its inverse. Thus, it is desirable to have estimation methods that do not require such quantities. Recently, Ene and Karahasan (2016) proposed one such order statistics-based method of estimation for estimating parameters of symmetric location-scale family.

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Let's summarize BLU (Best Linear Unbiased) estimation method based on order statistics for scale parameter σ. The notation of Balakrishnan and Cohen (1991) is adopted for the presentation of this study. Let $X_1, X_2, \ldots, X_n$ be a random sample from scale distribution with parameter σ. The observations of sample are ordered and then doubly Type II censored by taking the first r and last s ordered observations out of the sample,

$$X_{r+1:n} \leq X_{r+2:n} \leq \cdots \leq X_{n-s:n}, \tag{2}$$

where $X_{i:n}$ denotes ith order statistics in the sample of size n. By dividing order statistics in Eq. (2) by σ, $Z_{i:n} = X_{i:n}/\sigma$, the Type II censored sample of standardized Z's is found as,

$$Z_{r+1:n} \leq Z_{r+2:n} \leq \cdots \leq Z_{n-s:n}. \tag{3}$$

where $Z_{i:n}$ denotes ith standardized order statistics. Lloyd (1952) applied generalized Gauss-Markov theorem to the order statistics to obtain BLU estimators of location and scale parameters μ, σ (David and Nagaraja, 2003). With the notation of Balakrishnan and Cohen (1991), the BLU estimator of σ and its variance are defined as

$$\hat{\sigma} = \left(\frac{\alpha'\beta^{-1}}{\alpha'\beta^{-1}\alpha}\right)X = \sum_{i=r+1}^{n-s} a_i X_{i:n} \; Var(\hat{\sigma}) = \frac{\sigma^2}{\alpha'\beta^{-1}\alpha},$$

where α and β are the vector of expected values and matrix of variance-covariances for the order statistics of $Z's$ in Eq. (3), respectively. Further, **X** is the vector of the order statistics in (2) and $a_i's$ are constants.

Balakrishnan and Cohen (1991) and David and Nagaraja (2003) give the review of the literature about BLUE. In order to simplify the computation of BLU estimators various approaches have been proposed. These approaches can roughly be divided into three categories: those based on using simplified or approximate variance-covariance matrix of order statistics, the ones based on using weight function that depend on only distribution function and probability density function of observations, and those based on using selected order statistics.

Gupta's approach (1952) employs identity matrix I in place of variance-covariance matrix β, while Bloom's approach (1958, 1962) uses asymptotic approximations for the variances and covariances of the order statistics. Differently from the previous approaches, Downton (1966) proposes linear estimators with polynomial coefficients for location and scale parameters. Bennett (1952) and Jung (1955, 1962) reach asymptotically optimal linear unbiased estimators of location and scale parameters by making use of continuous weight functions. Because of dependence of the weight functions on only pdf's and cdf's, expected values, variances, and covariances of the order statistics are not necessary for this kind of approaches. Chan and Cheng (1988), and Ogawa (1951, 1952) discuss selecting k out of n order statistics among all ${}_nC_k$ combinations of order statistics so as to obtain BLU or asymptotically BLU estimators of location and scale based on selected order statistics (Balakrishnan and Cohen, 1991). Sarkadi (1985) proves the positivity of BLU estimators associated with scale parameter for distributions with log-concave density functions (David and Nagaraja, 2003).

Some BLU approaches make use of spacing of ordered observations or quasi ranges for the scale parameter of symmetric location scale family. Some of the works related to these approaches are Balakrishnan and Papadatos (2002), Sajeevkumar and Thomas (2010), Thomas (1990) and Sajeevkumar (2011). Further, there are some BLU estimation approaches based on various sampling schemes other than simple random sampling such as ranked set sampling, ordered ranked set sampling, etc. These include the works of Sinha and Purkayastha (1996), Tiwari and Kvam (2001), Hossain and Mutlak (2001), Zheng and Saleh (2003), Balakrishnan and Li (2005), and Shadid et al. (2011).

Since this paper deals with estimation in some scale family of distributions, let us review briefly some of the works on BLU estimation for some of these distributions. Balakrishnan and Wong (1994) extend the results of Balakrishnan and Puthenpura (1986) on BLU estimators of location scale parameters for half logistic distributions to singly and doubly Type II censored samples. Adatia (1997) gives an approximate BLU estimator of the scale parameter of half-logistic distribution based on a selected few order statistics.

While Dyer and Whisenand (1973) obtained the BLU estimator of scale parameter based on Type II censored samples of small size for Rayleigh distribution, Adatia (1995) obtained BLU estimators for moderately large censored samples. Akhter and Hırai (2009) compares efficiency of estimator of Bloom's (1958) with those of BLU and ML methods for the scale parameter of Rayleigh distribution. As for the exponential distribution, it is easy to reach BLU estimators in the cases of complete and censored samples for exponential distribution because expected values, variances and covariances of order statistics have explicit forms. Balakrishnan and Cohen (1991) give tables necessary for the asymptotically BLU estimator based on selected order statistics. Harter and Balakrishnan (1996) discuss the computation of accurate tables needed for simplified estimators based on one or on two order statistics for one parameter-exponential distribution.

By following essentially the ideas similar to those in the work of Ene and Karahasan (2016), new unbiased estimators based on the order statistics are reached for some families of scale distributions in the cases of uncensored and doubly Type II censored samples in this paper. Simulations are conducted to compare performance of the proposed estimators with those of BLU estimators.

The remainder of the paper proceeds as follows. The estimators proposed for some scale families of distributions are given in Section 2. Simulation settings are described in Section 3, and then the results of evaluations and comparisons from the simulation studies are presented. Next, some of these new estimators are applied to a real data set in Section 4. Finally, some discussions and concluding remarks are made in Section 5.

2. New estimators for some families of scale distributions

In this Section, two or three new unbiased estimators are proposed for each of the scale family of distributions considered in this paper. These new estimators are based on order statistics and can be computed for both uncensored and symmetrically doubly Type II censored samples. Further, they can be easily extended to Type II right or Type II left censored samples. Since the rationale behind the estimators is from the work of Ene and Karahasan (2016), the new estimators for scale parameter σ are called NE estimators.

The ideas leading to the new estimators are stated as follows. Let random variables $X_1, X_2, \ldots, X_n$ be a random sample from a scale distribution having probability function $f(x, \sigma)$ with parameter σ. Assume that the sample is symmetrically doubly Type II censored with the number of censored observations r and s equal on both side. In this case the sample is expressed in terms of order statistics as follows.

$$X_{r+1:n} \leq X_{r+2:n} \leq \cdots \leq X_{n-r:n}$$

Let $Z_i = X_i/\sigma$ be standardized random variable corresponding to X_i. Thus, the random variables $Z_1, Z_2, \ldots, Z_n$ can be regarded as a random sample from the distribution with probability density function $f_0(z)$, which is completely known, that is, free of scale parameter as indicated in Section 1.

$$f(x_i;\sigma) = \frac{1}{\sigma} f_0\left(\frac{x_i}{\sigma}\right) = \frac{1}{\sigma} f_0(z_i),$$

In fact, it is not possible to realize the sample $Z_1, Z_2, \ldots, Z_n$ corresponding to the sample $X_1, X_2, \ldots, X_n$ because the value of parameter σ is not known. Nevertheless, a random sample $Z_1, Z_2, \ldots, Z_n$ from the distribution with probability density function $f_0(z)$ can be obtained via simulation due to the fact that the distribution does not depend on σ. Note that the simulated random sample $Z_1, Z_2, \ldots, Z_n$ is independent of the sample $X_1, X_2, \ldots, X_n$. Then, the simulated random sample $Z_1, Z_2, \ldots, Z_n$ are ordered and symmetrically Type II censored as

$$Z_{r+1:n} \leq Z_{r+2:n} \leq \cdots \leq Z_{n-r:n}$$

where $Z_{i:n} = X_{i:n}/\sigma$, $i = r+1, \ldots, n-r$. Some of the new estimators that are introduced in this paper use $F_0\,(Z_{i:n})$ or some function of $F_0(Z_{i:n})$, $i = r+1, \ldots, n-r$ as weights. In order to determine the form of weights, a variety of functions of $F_0(Z_{i:n})$ are tried as weights by simulations for a given family of distributions. Then, the weights that approximately minimize standard error of these estimators are chosen as best. Empirical evidence obtained from the simulations show that some function of $F_0(Z_{i:n})$ as weight will do the best for constructing NE type estimators for the family of distribution in question. New estimators for the scale distributions considered are as follows.

2.1. Half normal distribution

$$\hat{\sigma}_{ne1} = \frac{\sum_{i=r+1}^{n-r} X_{i:n}Z_{i:n} + rX_{r+1:n}Z_{r+1:n} + rX_{n-r:n}Z_{n-r:n}}{\sum_{i=r+1}^{n-r} Z_{i:n} + rZ_{r+1:n} + rZ_{n-r:n}}$$

$$\hat{\sigma}_{ne2} = \frac{\sum_{i=r+1}^{n-r} X_{i:n}b_i + rX_{r+1:n}b_{r+1} + rX_{n-r:n}b_{n-r}}{\sum_{i=r+1}^{n-r} Z_{i:n}b_i + rZ_{r+1:n}b_{r+1} + rZ_{n-r:n}b_{n-r}}$$

$$\hat{\sigma}_{ne3} = \frac{\sum_{i=r+1}^{n-r} X_{i:n}b_i + rX_{r+1:n}b_{r+1} + rX_{n-r:n}b_{n-r}}{\sum_{i=r+1}^{n-r} b_i + rb_{r+1} + rb_{n-r}}$$

where the probability weights for $\hat{\sigma}_{ne2}$ and $\hat{\sigma}_{ne3}$ are defined as $b_i = F_0(Z_{i:n}) = P(Z \leq Z_{i:n})$, $i = r+1, \ldots, n-r$ and Z is the random variable with probability density function $f_0(z)$ of standard half normal distribution.

2.2. Half logistic distribution

$$\hat{\sigma}_{ne1} = \frac{\sum_{i=r+1}^{n-r} X_{i:n}Z_{i:n} + rX_{r+1:n}Z_{r+1:n} + rX_{n-r:n}Z_{n-r:n}}{\sum_{i=r+1}^{n-r} Z_{i:n} + rZ_{r+1:n} + rZ_{n-r:n}}$$

$$\hat{\sigma}_{ne2} = \frac{\sum_{i=r+1}^{n-r} X_{i:n}b_i + rX_{r+1:n}b_{r+1} + rX_{n-r:n}b_{n-r}}{\sum_{i=r+1}^{n-r} Z_{i:n}b_i + rZ_{r+1:n}b_{r+1} + rZ_{n-r:n}b_{n-r}}$$

$$\hat{\sigma}_{ne3} = \frac{\sum_{i=r+1}^{n-r} X_{i:n} b_i + rX_{r+1:n} b_{r+1} + rX_{n-r:n} b_{n-r}}{\sum_{i=r+1}^{n-r} b_i + rb_{r+1} + rb_{n-r}}$$

where the probability weights for the estimators $\hat{\sigma}_{ne2}$ and $\hat{\sigma}_{ne3}$ are defined as $b_i = 1.0 - F_0(Z_{i:n}) = 1.0 - P(Z \leq Z_{i:n})$, $i = r+1, \ldots, n-r$ and Z is the random variable with probability density function $f_0(z)$ of standard half logistic distribution. Indeed, instead of using the weights $b_i = 1.0 - F_0\,(Z_{i:n})$ for $\hat{\sigma}_{ne3}$, the weights $b_i^* = \exp[1.0 - F_0\,(Z_{i:n})]$, $i = r+1, \ldots, n-r$ can be used because both type weights give nearly identical standard error performances.

2.3. Log-weibull distribution

$$\hat{\sigma}_{ne1} = \frac{\sum_{i=r+1}^{n-r} |X_{i:n}|\, b_i + r\,|X_{r+1:n}|\, b_{r+1} + r\,|X_{n-r:n}|\, b_{n-r}}{\sum_{i=r+1}^{n-r} |Z_{i:n}|\, b_i + r\,|Z_{r+1:n}|\, b_{r+1} + r\,|Z_{n-r:n}|\, b_{n-r}}$$

$$\hat{\sigma}_{ne2} = \frac{\sum_{i=r+1}^{n-r} |X_{i:n}|\, b_i + r\,|X_{r+1:n}|\, b_{r+1} + r\,|X_{n-r:n}|\, b_{n-r}}{\sum_{i=r+1}^{n-r} b_i + rb_{r+1} + rb_{n-r}}$$

where the probability weights for $\hat{\sigma}_{ne1}$ and $\hat{\sigma}_{ne2}$ are defined as $b_i = F_0\,(Z_{i:n}) = P(Z \leq Z_{i:n})$, $i = r+1, \ldots, n-r$ and Z is the random variable with probability density function $f_0(z)$ of standard log-Weibull distribution.

2.4. Normal distribution

$$\hat{\sigma}_{ne1} = \frac{\sum_{i=r+1}^{n-r} |X_{i:n} Z_{i:n}| + r\,|X_{r+1:n} Z_{r+1:n}| + r\,|X_{n-r:n} Z_{n-r:n}|}{\sum_{i=r+1}^{n-r} |Z_{i:n}| + r\,|Z_{r+1:n}| + r\,|Z_{n-r:n}|}$$

$$\hat{\sigma}_{ne2} = \frac{\sum_{i=r+1}^{n-r} |X_{i:n}|\, b_i + r\,|X_{r+1:n}|\, b_{r+1} + r\,|X_{n-r:n}|\, b_{n-r}}{\sum_{i=r+1}^{n-r} |Z_{i:n}|\, b_i + r\,|Z_{r+1:n}|\, b_{r+1} + r\,|Z_{n-r:n}|\, b_{n-r}}$$

$$\hat{\sigma}_{ne3} = \frac{\sum_{i=r+1}^{n-r} |X_{i:n}|\, b_i + r\,|X_{r+1:n}|\, b_{r+1} + r\,|X_{n-r:n}|\, b_{n-r}}{\sum_{i=r+1}^{n-r} b_i + rb_{r+1} + rb_{n-r}}$$

where the probability weights for $\hat{\sigma}_{ne2}$ and $\hat{\sigma}_{ne3}$ are defined as

$$b_i = \begin{cases} P(Z_{i:n} \leq Z \leq 0), & Z_{i:n} \leq 0 \\ P(0 \leq Z < Z_{i:n}), & Z_{i:n} > 0 \end{cases} \qquad i = r+1, \ldots, n-r,$$

and Z is the random variable with probability density function $f_0(z)$ of standard normal distribution.

2.5. Exponential distribution

$$\hat{\sigma}_{ne1} = \frac{\sum_{i=r+1}^{n-r} X_{i:n} Z_{i:n} + rX_{r+1:n} Z_{r+1:n} + rX_{n-r:n} Z_{n-r:n}}{\sum_{i=r+1}^{n-r} Z_{i:n} + rZ_{r+1:n} + rZ_{n-r:n}}$$

$$\hat{\sigma}_{ne2} = \frac{\sum_{i=r+1}^{n-r} X_{i:n} + rX_{r+1:n} + rX_{n-r:n}}{\sum_{i=r+1}^{n-r} Z_{i:n} + rZ_{r+1:n} + rZ_{n-r:n}}$$

$$\hat{\sigma}_{ne3} = \frac{\sum_{i=r+1}^{n-r} X_{i:n}b_i + rX_{r+1:n}b_{r+1} + rX_{n-r:n}b_{n-r:}}{\sum_{i=r+1}^{n-r} b_i + rb_{r+1} + rb_{n-r}}$$

where Z is the random variable with probability density function $f_0(z)$ of standard exponential distribution and weights for $\hat{\sigma}_{ne3}$ are defined as $b_i = exp\left[F_0\left(Z_{i:n}\right)\right] = exp\left[P(Z \leq Z_{i:n})\right]$, $i = r+1, \ldots, n-r$. In fact, instead of using the weights $b_i = exp\left[F_0\left(Z_{i:n}\right)\right]$, the following weights $b_i^* = F_0\left(Z_{i:n}\right)$, $i = r+1, \ldots, n-r$ can be used because both type weights give almost the same standard error performance. The estimator $\hat{\sigma}_{ne2}$ need not be expressed in terms of order statistics for the case of uncensored samples.

2.6. Rayleigh distribution

$$\hat{\sigma}_{ne1} = \frac{\sum_{i=r+1}^{n-r} X_{i:n}Z_{i:n} + rX_{r+1:n}Z_{r+1:n} + rX_{n-r:n}Z_{n-r:n}}{\sum_{i=r+1}^{n-r} Z_{i:n} + rZ_{r+1:n} + rZ_{n-r:n}}$$

$$\hat{\sigma}_{ne2} = \frac{\sum_{i=r+1}^{n-r} X_{i:n}b_i + rX_{r+1:n}b_{r+1} + rX_{n-r:n}b_{n-r}}{\sum_{i=r+1}^{n-r} Z_{i:n}b_i + rZ_{r+1:n}b_{r+1} + rZ_{n-r:n}b_{n-r}}$$

$$\hat{\sigma}_{ne3} = \frac{\sum_{i=r+1}^{n-r} X_{i:n}b_i + rX_{r+1:n}b_{r+1} + rX_{n-r:n}b_{n-r}}{\sum_{i=r+1}^{n-r} b_i + rb_{r+1} + rb_{n-r}}$$

where the probability weights for $\hat{\sigma}_{ne2}$ and $\hat{\sigma}_{ne3}$ are defined as $b_i = F_0\left(Z_{i:n}\right) = P(Z \leq Z_{i:n})$, $i = r+1, \ldots, n-r$ and Z is the random variable with probability density function $f_0(z)$ of standard Rayleigh distribution.

2.7. Uniform distribution

$$\hat{\sigma}_{ne1} = \frac{\sum_{i=r+1}^{n-r} X_{i:n}b_i + rX_{r+1:n}b_{r+1} + rX_{n-r:n}b_{n-r}}{\sum_{i=r+1}^{n-r} Z_{i:n}b_i + rZ_{r+1:n}b_{r+1} + rZ_{n-r:n}b_{n-r}}$$

$$\hat{\sigma}_{ne2} = \frac{\sum_{i=r+1}^{n-r} X_{i:n}b_i + rX_{r+1:n}b_{r+1} + rX_{n-r:n}b_{n-r}}{\sum_{i=r+1}^{n-r} b_i + rb_{r+1} + rb_{n-r}}$$

where the probability weights for $\hat{\sigma}_{ne1}$ and $\hat{\sigma}_{ne2}$ are defined as $b_i = F_0(Z_{i:n})^{50} = P(Z \leq Z_{i:n})^{50}$, $i = r+1, \ldots, n-r$ and Z is the random variable with probability density function $f_0(z)$ of standard uniform distribution. The empirical evidence obtained from the simulations that is not shown here suggests that as the exponent of $F_0\left(Z_{i:n}\right)$ increases, first the standard error performance of NE estimator improves, and then it stabilizes around a value. Thus, exponent 50 is not a magic number; it is just large enough for $\hat{\sigma}_{ne1}$ and $\hat{\sigma}_{ne2}$ to show better performance. For example, it might be taken as 30 or 40 which does not change the performance of NE estimator substantially.

Note that all the NE estimators that use the weights $b_i = F_0\left(Z_{i:n}\right)$ or some function of it can be reformulated by replacing $F_0\left(Z_{i:n}\right)$ with $U_{i:n}$, the ith order statistics of a random sample size of n from uniform (0, 1) distribution. The reason for this is distributional equivalence of $F_0\left(Z_{i:n}\right)$ and $U_{i:n}$ due to the Probability Transformation. Such a reformulation is useful in computation since it is not necessary to take numerical integration for calculating $F_0\left(Z_{i:n}\right)$ for the distribution functions that cannot be expressed in a closed form; all that is needed is to simulate the order statistics $U_{i:n}$, $i = 1, 2, \ldots, n$. and use them as weights.

All the estimators become suitable for uncensored samples when the censoring level is taken to be zero, i.e., $r = 0$ in the formula of these estimators. Although the new estimators are different in nature for some of these distributions, they are denoted as NE1, NE2 and NE3 when three proposals are made, and as NE1, NE2 when only two proposals are made for simplicity in notation. These new estimators are defined by using only a single doubly Type II censored random sample $Z_1, Z_2, \ldots, Z_n$ simulated from the relative distribution with probability density function $f_0(z)$. Some improvement can be achieved in these estimators by simulating more than one such $Z_1, Z_2, \ldots, Z_n$ independent sample and computing estimators from each sample, say k samples, and then taking average of these k estimators as in Eq. (4).

$$\hat{\sigma}_{ne1}^{k} = \varphi_{n,r}\left(\frac{\sum_{j=1}^{k}\hat{\sigma}_{ne1,j}}{k}\right)\hat{\sigma}_{ne2}^{k} = \phi_{n,r}\left(\frac{\sum_{j=1}^{k}\hat{\sigma}_{ne2,j}}{k}\right)\hat{\sigma}_{ne3}^{k} = \tau_{n,r}\left(\frac{\sum_{j=1}^{k}\hat{\sigma}_{ne3,j}}{k}\right) \tag{4}$$

where $\varphi_{n,r}$, $\phi_{n,r}$ and, $\tau_{n,r}$ are the constants that make the estimators $\hat{\sigma}_{ne1}^{k}$, $\hat{\sigma}_{ne2}^{k}$ and, $\hat{\sigma}_{ne3}^{k}$ respectively unbiased. In the simulations, it is observed that the constant changes only according to distribution of population, sample size and censoring level. Tables 1–3 present values of the constants computed from simulations of 50,000 repetitions. Moreover, $\hat{\sigma}_{ne1}^{k}$, $\hat{\sigma}_{ne2}^{k}$ and, $\hat{\sigma}_{ne3}^{k}$ are not linear estimators due to either randomness of the weights b_i's or their involving cross productions of the order statistics $X_{i:n}$ and $Z_{i:n}$.

Further, relevant estimators of standard deviations for the estimators $\hat{\sigma}_{ne1}^{k}$, $\hat{\sigma}_{ne2}^{k}$ and, $\hat{\sigma}_{ne3}^{k}$ are proposed. These estimators are defined in the form of sample standard deviations of k

Table 1. The multiplying constants $\varphi_{n,r}$ that make scale estimators $\hat{\sigma}_{ne1}^{k}$ unbiased for every value of k and parameters.

		half normal	half logistic	log-Weibull	Normal	Exponential	Rayleigh	Uniform
$n=5$	$r=0$	0.9173	0.5087	0.7616	1.0802	0.6485	0.6770	0.9704
	$r=1$	1.1880	0.6982	0.6634	1.7255	0.9815	0.7769	0.9066
$n=10$	$r=0$	0.8642	0.4705	0.8626	0.9596	0.5852	0.6553	0.9996
	$r=1$	0.9846	0.5627	0.8478	1.2419	0.7442	0.7057	0.9853
	$r=2$	1.1326	0.6644	0.8161	1.6764	0.9259	0.7572	0.9665
	$r=3$	1.2936	0.7735	0.7273	2.3356	1.1370	0.8057	0.9466
$n=15$	$r=0$	0.8434	0.4576	0.9018	0.9118	0.5622	0.6471	1.0055
	$r=1$	0.9203	0.5197	0.8971	1.0785	0.6696	0.6806	0.9968
	$r=2$	1.0101	0.5821	0.8896	1.3009	0.7803	0.7155	0.9881
	$r=3$	1.1095	0.6511	0.8753	1.6107	0.9045	0.7504	0.9814
	$r=0$	0.8337	0.4509	0.9258	0.8876	0.5481	0.6417	1.0067
$n=20$	$r=1$	0.8888	0.4971	0.9222	1.0049	0.6312	0.6667	1.0007
	$r=2$	0.9510	0.5432	0.9163	1.1487	0.7102	0.6927	0.9962
	$r=3$	1.0205	0.5920	0.9117	1.3276	0.7978	0.7199	0.9912

Table 2. The multiplying constants $\phi_{n,r}$ that make scale estimators $\hat{\sigma}^k_{ne2}$ unbiased for every value of k and parameters.

		half normal	half logistic	Normal	Exponential	Rayleigh	log-Weibull	Uniform
$n=5$	$r=0$	0.9234	0.8539	1.0643	0.8008	0.9785	1.2947	1.2126
	$r=1$	0.8642	0.8108	0.9545	0.7477	0.9555	1.5552	1.5109
$n=10$	$r=0$	0.9731	0.9298	1.0639	0.8994	0.9944	1.3803	1.1143
	$r=1$	0.9584	0.9217	1.0749	0.8893	0.9907	1.5237	1.2319
	$r=2$	0.9451	0.9077	1.0880	0.8751	0.9845	1.7374	1.3825
	$r=3$	0.9262	0.8910	1.0279	0.8569	0.9748	1.9548	1.5786
$n=15$	$r=0$	0.9828	0.9529	1.0505	0.9338	0.9972	1.4180	1.0813
	$r=1$	0.9781	0.9507	1.0547	0.9304	0.9956	1.5070	1.1520
	$r=2$	0.9730	0.9462	1.0677	0.9210	0.9934	1.6344	1.2386
	$r=3$	0.9662	0.9396	1.0860	0.9175	0.9922	1.8045	1.3399
	$r=0$	0.9877	0.9649	1.0426	0.9483	0.9982	1.4388	1.0651
$n=20$	$r=1$	0.9857	0.9636	1.0430	0.9466	0.9972	1.4957	1.1152
	$r=2$	0.9838	0.9611	1.0496	0.9437	0.9965	1.5797	1.1745
	$r=3$	0.9812	0.9579	1.0577	0.9419	0.9955	1.6949	1.2424

estimators as in Eq. (5).

$$S_{\hat{\sigma}^k_{ne1}} = \gamma_{n,r}\phi_{n,r}\sqrt{\frac{\sum_{i=1}^{k}\left(\hat{\sigma}_{ne1,i}-\bar{\hat{\sigma}}_{ne1}\right)^2}{k-1}} \quad S_{\hat{\sigma}^k_{ne2}} = \kappa_{n,r}\,\phi_{n,r}\sqrt{\frac{\sum_{i=1}^{k}\left(\hat{\sigma}_{ne2,i}-\bar{\hat{\sigma}}_{ne2}\right)^2}{k-1}}$$
$$S_{\hat{\sigma}^k_{ne3}} = \nu_{n,r}\,\tau_{n,r}\sqrt{\frac{\sum_{i=1}^{k}\left(\hat{\sigma}_{ne3,i}-\bar{\hat{\sigma}}_{ne3}\right)^2}{k-1}} \tag{5}$$

Tables 4–6 display the values of the constants $\gamma_{n,r}$, $\kappa_{n,r}$ and, $\upsilon_{n,r}$ which are determined by simulations with 50,000 repetitions. These constants make these estimators unbiased and rely on only population distribution, sample size and censoring degree in general. However, they may also slightly decrease with large values of k, say 1,000, in some cases such as half logistic and exponential distributions. Furthermore, it is important that the constants in Tables 2, 3, 5, and 6 be modified if $U_{i:n},\ i = 1, 2, \ldots, n$ are used in computation of the estimators instead of $F_0\left(Z_{i:n}\right),\ i = 1, 2, \ldots, n$.

Table 3. The multiplying constants $\tau_{n,r}$ that make scale estimators $\hat{\sigma}^k_{ne3}$ unbiased for every value of k and parameters.

		half normal	half logistic	Normal	Exponential	Rayleigh
$n=5$	$r=0$	0.9668	0.7125	1.1104	0.8567	0.6600
	$r=1$	1.2078	0.8000	1.7275	1.1028	0.7624
$n=10$	$r=0$	0.9260	0.7108	1.0137	0.8263	0.6398
	$r=1$	1.0273	0.7596	1.2653	0.9494	0.6839
	$r=2$	1.1584	0.8058	1.6834	1.0899	0.7385
	$r=3$	1.3083	0.8483	2.3318	1.2274	0.7941
$n=15$	$r=0$	0.9127	0.7080	0.9766	0.8159	0.6319
	$r=1$	0.9715	0.7424	1.1171	0.8949	0.6585
	$r=2$	1.0485	0.7770	1.3226	0.9813	0.6927
	$r=3$	1.1397	0.8094	1.6205	1.0798	0.7306
	$r=0$	0.9073	0.7075	0.9566	0.8105	0.6283
$n=20$	$r=1$	0.9468	0.7344	1.0499	0.8658	0.6470
	$r=2$	0.9990	0.7595	1.1764	0.9301	0.6702
	$r=3$	1.0605	0.7845	1.3479	1.0011	0.6968

Table 4. The multiplying constants $\gamma_{n,r}$ that make the estimator $S_{\hat{\sigma}^k_{ne1}}$ unbiased for every value of k and parameters*.

		half normal	half logistic	log-Weibull	Normal	Exponential	Rayleigh	Uniform
$n=5$	$r=0$	3.5516	3.7371	0.8652	1.9665	3.8770	4.2320	0.7362
	$r=1$	5.9589	6.0985	0.8084	2.0622	6.2489	7.9830	0.6948
$n=10$	$r=0$	3.1330	3.2062	0.9324	2.1188	3.1735	3.8042	0.8586
	$r=1$	3.8041	4.0276	0.9396	2.0707	4.2186	4.5619	0.8559
	$r=2$	4.7636	4.9482	0.9307	1.7852	5.1365	6.2043	0.8408
	$r=3$	7.5629	7.5320	0.8799	1.7950	7.3468	10.9514	0.8229
$n=15$	$r=0$	2.9981	3.0046	0.9580	2.1940	2.9550	3.6320	0.9091
	$r=1$	3.4551	3.6265	0.9589	2.3257	3.7204	4.1063	0.9074
	$r=2$	3.8522	4.0913	0.9641	2.2073	4.3287	4.7142	0.9068
	$r=3$	4.3994	4.6248	0.9625	1.9481	4.8479	5.7401	0.8999
	$r=0$	2.9318	2.9020	0.9745	2.2875	2.8332	3.5554	0.9353
$n=20$	$r=1$	3.2883	3.3857	0.9767	2.4848	3.4653	3.8824	0.9424
	$r=2$	3.5994	3.7850	0.9803	2.4890	3.9515	4.2487	0.9326
	$r=3$	3.8489	4.1151	0.9776	2.3695	4.3490	4.7710	0.9419

* The constants may decrease slightly for some distributions as values of k become larger.

Table 5. The multiplying constants $\kappa_{n,r}$ that make the estimator $S_{\hat{\sigma}^k_{ne2}}$ unbiased for every value of k and parameters*.

		half normal	half logistic	Normal	Exponential	Rayleigh	log-Weibull	Uniform
$n=5$	$r=0$	0.8415	0.8619	0.8144	0.8491	0.9357	2.9311	7.1055
	$r=1$	0.7780	0.8021	0.6756	0.8043	0.9114	5.0316	16.2019
$n=10$	$r=0$	0.9424	0.9418	0.9271	0.9353	0.9864	2.8651	4.8033
	$r=1$	0.9228	0.9307	0.8586	0.9267	0.9705	2.9601	10.8728
	$r=2$	0.9011	0.9193	0.7750	0.9137	0.9714	3.6400	18.4747
	$r=3$	0.8852	0.8927	0.7121	0.9045	0.9588	6.6016	26.1570
$n=15$	$r=0$	0.9768	0.9748	0.9528	0.9703	1.0019	2.8877	3.9525
	$r=1$	0.9605	0.9710	0.9306	0.9625	0.9949	2.8571	8.3300
	$r=2$	0.9585	0.9604	0.8850	0.9568	0.9904	2.9674	13.8718
	$r=3$	0.9484	0.9513	0.8289	0.9563	0.9896	3.2936	20.2341
	$r=0$	0.9873	0.9927	0.9742	0.9814	1.0127	2.9355	3.4228
$n=20$	$r=1$	0.9792	0.9847	0.9592	0.9774	1.0075	2.8771	6.9053
	$r=2$	0.9746	0.9795	0.9454	0.9775	0.9968	2.8994	11.3341
	$r=3$	0.9673	0.9717	0.9116	0.9721	1.0032	2.9466	16.3957

* The constants may decrease slightly for some distributions as values of k become larger.

Table 6. The multiplying constants $\nu_{n,r}$ that make the estimator $S_{\hat{\sigma}^k_{ne3}}$ unbiased for every value of k and parameters*.

		half normal	half logistic	Normal	Exponential	Rayleigh
$n=5$	$r=0$	3.6547	19.7728	2.2523	9.9897	3.4393
	$r=1$	6.1048	37.3033	2.1925	15.4352	5.8381
$n=10$	$r=0$	3.3573	17.4797	2.3766	9.6904	3.1493
	$r=1$	3.8329	17.9804	2.2805	10.3218	3.5894
	$r=2$	4.8038	24.2633	1.9272	12.1555	4.5146
	$r=3$	7.9205	48.7313	1.8728	19.2493	7.6556
$n=15$	$r=0$	3.2515	17.0281	2.4634	9.7123	3.0793
	$r=1$	3.5219	16.1927	2.5538	9.9069	3.2744
	$r=2$	3.8355	17.3502	2.4136	10.3817	3.6208
	$r=3$	4.4570	21.1123	2.0717	11.4152	4.1865
$n=20$	$r=0$	3.2274	16.7962	2.5405	9.8121	3.0377
	$r=1$	3.4025	15.8182	2.6836	9.8439	3.2030
	$r=2$	3.5973	15.7539	2.7010	9.9571	3.3591
	$r=3$	3.8589	16.9485	2.5459	10.3367	3.6087

Table 7. Expected values, standard deviations, absolute biases and rmse values and efficiencies associated with BLU and NE estimators for half normal ($\sigma = 3$) distributed data with 50,000 repetitions.

$n = 5$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	3.0065	0.9747	0.0065	0.9747	1.0000			
NE1	3.0040	0.9753	0.0040	0.9753	0.9994	0.9738	0.0015	0.4759
NE2	3.0061	1.0064	0.0061	1.0064	0.9685	0.9981	0.0083	0.4803
NE3	3.0076	0.9789	0.0076	0.9790	0.9957	0.9732	0.0057	0.4505
$r = 1$								
BLUE	3.0039	1.1396	0.0039	1.1396	1.0000			
NE1	2.9998	1.1409	0.0002	1.1409	0.9989	1.1447	0.0037	0.7216
NE2	2.9999	1.1923	0.0001	1.1923	0.9558	1.1876	0.0047	0.7992
NE3	2.9993	1.1427	0.0007	1.1427	0.9973	1.1436	0.0009	0.7198

EFF(RMSE$(\hat{\sigma})$ denotes efficiency of estimators compared to BLU estimators.

3. Simulation study

The samples of size $n = 5,\ 10,\ 15,\ 20$ have been generated as either uncensored or symmetric doubly Type II censored with censoring degrees of $r = 1,\ 2,\ 3$ from half normal (3), one-parameter half logistic (3), log-Weibull (0, 3), normal (0, 3), one-parameter exponential (3), one-parameter Rayleigh (3) and uniform (0, 3) distributions with 50,000 repetitions. The required vector of expected values and variance-covariance matrix of order statistics for BLU estimation are obtained from Harter and Balakrishnan (1996) and by means of simulations from relevant distributions with 1,000,000 repetitions. All the computations have been performed through a computer program of Java codes specially developed for this study.

The estimation results for the half normal distribution are given in Tables 7–10; for the half logistic distribution in Tables 11–14; for the log-Weibull distribution in Tables 15–18; for the normal distribution in Tables 19–22; for the exponential distribution in Tables 23–26; for the Rayleigh distribution in Tables 27–30, and for the uniform distribution in Tables 31–34. In these tables NE1 indicates the estimator $\hat{\sigma}^k_{ne1}$, NE2 the estimator $\hat{\sigma}^k_{ne2}$ and, NE3 the estimator $\hat{\sigma}^k_{ne3}$. The tables give values of expectations, absolute biases, standard deviations,

Table 8. Expected values, standard deviations, absolute biases and rmse values and efficiencies associated with BLU and NE estimators for half normal ($\sigma = 3$) distributed data with 50,000 repetitions.

$n = 10$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	3.0036	0.6901	0.0036	0.6901	1.0000			
NE1	3.0052	0.6825	0.0052	0.6825	1.0111	0.6811	0.0014	0.2426
NE2	3.0074	0.6984	0.0074	0.6984	0.9881	0.7003	0.0019	0.2209
NE3	2.9999	0.6847	0.0001	0.6847	1.0079	0.6846	0.0001	0.2241
$r = 1$								
BLUE	3.0061	0.7357	0.0061	0.7357	1.0000			
NE1	3.0056	0.7369	0.0056	0.7370	0.9984	0.7387	0.0018	0.2678
NE2	2.9998	0.7533	0.0002	0.7533	0.9766	0.7556	0.0023	0.2569
NE3	3.0082	0.7392	0.0082	0.7392	0.9953	0.7420	0.0028	0.2593
$r = 2$								
BLUE	3.0020	0.8008	0.0020	0.8008	1.0000			
NE1	3.0042	0.8036	0.0042	0.8036	0.9965	0.8088	0.0052	0.3321
NE2	2.9985	0.8215	0.0015	0.8215	0.9748	0.8193	0.0022	0.3013
NE3	2.9982	0.8036	0.0018	0.8036	0.9965	0.8069	0.0033	0.3285
$r = 3$								
BLUE	3.0019	0.8829	0.0019	0.8829	1.0000			
NE1	3.0059	0.8900	0.0059	0.8900	0.9920	0.8911	0.0010	0.4761
NE2	3.0067	0.9137	0.0067	0.9137	0.9663	0.9093	0.0044	0.3671
NE3	3.0054	0.8918	0.0054	0.8918	0.9900	0.8889	0.0029	0.4769

Table 9. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for half normal ($\sigma = 3$) distributed data with 50,000 repetitions.

$n = 15$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	3.0025	0.5521	0.0025	0.5521	1.0000			
NE1	3.0015	0.5531	0.0015	0.5531	0.9982	0.5533	0.0002	0.1663
NE2	3.0008	0.5665	0.0008	0.5665	0.9746	0.5694	0.0029	0.1491
NE3	3.0000	0.5564	0.0000	0.5564	0.9923	0.5563	0.0001	0.1528
$r = 1$								
BLUE	2.9999	0.5862	0.0001	0.5862	1.0000			
NE1	2.9992	0.5870	0.0008	0.5870	0.9986	0.5854	0.0016	0.1771
NE2	2.9973	0.5998	0.0027	0.5998	0.9773	0.5936	0.0062	0.1628
NE3	2.9982	0.5884	0.0018	0.5884	0.9963	0.5854	0.0030	0.1679
$r = 2$								
BLUE	3.0029	0.6174	0.0029	0.6174	1.0000			
NE1	3.0061	0.6190	0.0061	0.6191	0.9974	0.6169	0.0021	0.1913
NE2	3.0008	0.6302	0.0003	0.6302	0.9797	0.6331	0.0029	0.1804
NE3	3.0017	0.6195	0.0017	0.6195	0.9966	0.6154	0.0041	0.1853
$r = 3$								
BLUE	2.9986	0.6527	0.0014	0.6527	1.0000			
NE1	2.9988	0.6542	0.0012	0.6542	0.9977	0.6494	0.0048	0.2192
NE2	2.9967	0.6674	0.0033	0.6674	0.9780	0.6709	0.0035	0.2019
NE3	2.9971	0.6551	0.0029	0.6551	0.9963	0.6568	0.0017	0.2190

rmse's (root mean square error) and, efficiencies with respect to BLU estimator computed in simulations with 50,000 repetitions for the proposed estimators. In addition, the simulation results associated with standard deviation estimators $S_{\hat{\sigma}^k_{ne1}}$, $S_{\hat{\sigma}^k_{ne2}}$ and $S_{\hat{\sigma}^k_{ne3}}$ of $\hat{\sigma}^k_{ne1}$ $\hat{\sigma}^k_{ne2}$, and, $\hat{\sigma}^k_{ne3}$, respectively, are displayed in these tables.

The results in these tables confirm that the new estimators $\hat{\sigma}^k_{ne1}$, $\hat{\sigma}^k_{ne2}$, and $\hat{\sigma}^k_{ne3}$ proposed for the relevant family of scale distributions and their standard deviation estimators $S_{\hat{\sigma}^k_{ne1}}$, $S_{\hat{\sigma}^k_{ne2}}$ and, $S_{\hat{\sigma}^k_{ne3}}$ are unbiased as expected. As for the standard deviation performances, the performances associated with all or some of the estimators $\hat{\sigma}^k_{ne1}$, $\hat{\sigma}^k_{ne2}$ and, $\hat{\sigma}^k_{ne3}$ are in general either

Table 10. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for half normal ($\sigma = 3$) distributed data with 50,000 repetitions.

$n = 20$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	3.0004	0.4784	0.0004	0.4784	1.0000			
NE1	3.0013	0.4798	0.0013	0.4798	0.9971	0.4789	0.0009	0.1276
NE2	2.9987	0.4907	0.0013	0.4907	0.9749	0.4887	0.0020	0.1133
NE3	3.0019	0.4834	0.0019	0.4834	0.9897	0.4847	0.0013	0.1186
$r = 1$								
BLUE	2.9981	0.4997	0.0019	0.4997	1.0000			
NE1	2.9958	0.5003	0.0042	0.5004	0.9988	0.4986	0.0017	0.1344
NE2	2.9978	0.5113	0.0022	0.5113	0.9773	0.5063	0.0050	0.1217
NE3	2.9976	0.5024	0.0024	0.5024	0.9946	0.4998	0.0027	0.1268
$r = 2$								
BLUE	3.0009	0.5212	0.0009	0.5212	1.0000			
NE1	2.9986	0.5217	0.0014	0.5217	0.9990	0.5229	0.0012	0.1424
NE2	3.0032	0.5328	0.0032	0.5328	0.9782	0.5289	0.0039	0.1318
NE3	3.0020	0.5237	0.0020	0.5237	0.9952	0.5206	0.0030	0.1357
$r = 3$								
BLUE	2.9992	0.5401	0.0008	0.5401	1.0000			
NE1	2.9974	0.5405	0.0026	0.5405	0.9993	0.5374	0.0031	0.1498
NE2	3.0032	0.5528	0.0032	0.5528	0.9770	0.5504	0.0024	0.1400
NE3	3.0002	0.5421	0.0002	0.5421	0.9963	0.5427	0.0006	0.1474

Table 11. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for half logistic $(\sigma = 3)$ distributed data with 50,000 repetitions.

$n=5$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r=0$								
BLUE	2.9981	1.1241	0.0019	1.1241	1.0000			
NE1	2.9996	1.1460	0.0004	1.1460	0.9809	1.1533	0.0073	0.6611
NE2	3.0027	1.1325	0.0027	1.1325	0.9926	1.1375	0.0050	0.4410
NE3	2.9966	1.1288	0.0034	1.1288	0.9958	1.1323	0.0035	0.5291
$r=1$								
BLUE	3.0006	1.2592	0.0006	1.2592	1.0000			
NE1	2.9962	1.2584	0.0038	1.2584	1.0006	1.2545	0.0039	0.8333
NE2	3.0088	1.2774	0.0088	1.2775	0.9858	1.2728	0.0047	0.5649
NE3	2.9967	1.2698	0.0033	1.2698	0.9917	1.2678	0.0020	0.8275

very close or nearly equivalent to that of BLU estimator. This can be seen from efficiencies of these estimators compared to BLU estimators. Further, in some cases especially for the case of normal distribution, some NE estimators show slightly better performance than BLU estimators. The success of the estimators $\hat{\sigma}^k_{ne1}$, $\hat{\sigma}^k_{ne2}$ and, $\hat{\sigma}^k_{ne3}$ is noticeable mostly for normal, Rayleigh, exponential, half normal, half logistic distributions.

Moreover, sampling distributions of size 50,000 for the new estimators have been formed and their normality tested mostly with tests of Anderson and Darling (1954). Also test of Ryan and Joiner (1976) and Kolmogorov-Smirnov tests by Kolmogorov (1933) and Smirnov (1948) are used for normality in some cases. It has been, however, seen that these sampling distributions are not normal distributions. Only some power transformations of them in the range of (0, 1) do seem normally distributed for all the distributions but the uniform distribution (See Tables 35–41). Using Taylor series approximation for approximate mean and variances, approximate sampling distributions of $(\hat{\sigma}^k_{ne1})^{c_1}$, $(\hat{\sigma}^k_{ne2})^{c_2}$, and $(\hat{\sigma}^k_{ne3})^{c_3}$ will be expressed as

$$\left(\hat{\sigma}^k_{ne1}\right)^{c_1} \sim N\left(\sigma^{c_1},\ c_1^2\sigma^{2(c_1-1)}Var(\hat{\sigma}^k_{ne1})\right) \quad \left(\hat{\sigma}^k_{ne2}\right)^{c_2} \sim N\left(\sigma^{c_2},\ c_2^2\sigma^{2(c_2-1)}Var(\hat{\sigma}^k_{ne2})\right)$$

$$\left(\hat{\sigma}^k_{ne3}\right)^{c_3} \sim N\left(\sigma^{c_3},\ c_3^2\sigma^{2(c_3-1)}Var(\hat{\sigma}^k_{ne3})\right)$$

Table 12. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for half logistic $(\sigma = 3)$ distributed data with 50,000 repetitions.

$n=10$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r=0$								
BLUE	2.9991	0.7993	0.0009	0.7993	1.0000			
NE1	2.9933	0.8207	0.0067	0.8207	0.9739	0.8180	0.0027	0.3542
NE2	3.0014	0.8050	0.0014	0.8050	0.9929	0.8023	0.0027	0.2187
NE3	3.0047	0.8051	0.0047	0.8052	0.9928	0.8007	0.0044	0.2510
$r=1$								
BLUE	2.9962	0.8309	0.0038	0.8309	1.0000			
NE1	2.9948	0.8385	0.0052	0.8386	0.9909	0.8366	0.0019	0.3328
NE2	3.0020	0.8406	0.0020	0.8406	0.9885	0.8362	0.0044	0.2384
NE3	2.9946	0.8376	0.0054	0.8376	0.9920	0.8377	0.0002	0.2897
$r=2$								
BLUE	3.0028	0.8847	0.0028	0.8847	1.0000			
NE1	3.0011	0.8862	0.0011	0.8862	0.9983	0.8827	0.0035	0.3788
NE2	3.0019	0.8962	0.0019	0.8962	0.9872	0.8960	0.0002	0.2726
NE3	2.9991	0.8943	0.0009	0.8943	0.9893	0.8896	0.0047	0.3660
$r=3$								
BLUE	2.9998	0.9433	0.0002	0.9433	1.0000			
NE1	2.9970	0.9434	0.0030	0.9434	0.9999	0.9467	0.0033	0.5098
NE2	2.9948	0.9576	0.0052	0.9576	0.9851	0.9633	0.0057	0.3143
NE3	3.0062	0.9600	0.0062	0.9600	0.9826	0.9615	0.0015	0.5149

Table 13. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for half logistic ($\sigma = 3$) distributed data with 50,000 repetitions.

$n = 15$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	2.9962	0.6532	0.0038	0.6532	1.0000			
NE1	2.9974	0.6733	0.0026	0.6733	0.9701	0.6673	0.0060	0.2440
NE2	2.9938	0.6519	0.0062	0.6520	1.0020	0.6473	0.0046	0.1431
NE3	2.9962	0.6520	0.0038	0.6520	1.0018	0.6525	0.0005	0.1644
$r = 1$								
BLUE	2.9997	0.6680	0.0003	0.6680	1.0000			
NE1	3.0048	0.6751	0.0048	0.6751	0.9895	0.6821	0.0070	0.2248
NE2	3.0029	0.6686	0.0029	0.6687	1.0290	0.6768	0.0082	0.1535
NE3	2.9983	0.6670	0.0017	0.6670	1.0015	0.6713	0.0043	0.1774
$r = 2$								
BLUE	2.9953	0.6919	0.0047	0.6919	1.0000			
NE1	2.9948	0.6963	0.0052	0.6963	0.9937	0.6919	0.0044	0.2256
NE2	2.9995	0.6991	0.0005	0.6991	0.9897	0.6962	0.0029	0.1653
NE3	2.9976	0.6981	0.0024	0.6981	0.9911	0.6983	0.0002	0.2024
$r = 3$								
BLUE	3.0010	0.7232	0.0010	0.7232	1.0000			
NE1	2.9979	0.7239	0.0021	0.7239	0.9990	0.7181	0.0058	0.2461
NE2	3.0003	0.7336	0.0003	0.7336	0.9858	0.7288	0.0048	0.1821
NE3	3.0033	0.7333	0.0033	0.7334	0.9862	0.7290	0.0043	0.2373

Table 14. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for half logistic ($\sigma = 3$) distributed data with 50,000 repetitions.

$n = 20$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	2.9975	0.5627	0.0025	0.5627	1.0000			
NE1	3.0020	0.5846	0.0020	0.5846	0.9625	0.5914	0.0068	0.1919
NE2	2.9957	0.5655	0.0043	0.5655	0.9950	0.5620	0.0035	0.1079
NE3	2.9990	0.5658	0.0010	0.5658	0.9945	0.5622	0.0036	0.1219
$r = 1$								
BLUE	2.9994	0.5751	0.0006	0.5751	1.0000			
NE1	3.0004	0.5871	0.0004	0.5871	0.9796	0.5877	0.0006	0.1724
NE2	3.0001	0.5795	0.0001	0.5795	0.9924	0.5742	0.0053	0.1129
NE3	3.0020	0.5793	0.0020	0.5793	0.9927	0.5759	0.0034	0.1289
$r = 2$								
BLUE	2.9976	0.5865	0.0024	0.5865	1.0000			
NE1	2.9980	0.5926	0.0020	0.5926	0.9897	0.6018	0.0092	0.1688
NE2	2.9976	0.5927	0.0024	0.5927	0.9895	0.5924	0.0002	0.1195
NE3	2.9960	0.5917	0.0040	0.5918	0.9912	0.5894	0.0023	0.1383
$r = 3$								
BLUE	3.0032	0.6059	0.0032	0.6059	1.0000			
NE1	3.0026	0.6093	0.0026	0.6093	0.9944	0.6151	0.0058	0.1731
NE2	3.0029	0.6131	0.0029	0.6131	0.9882	0.6104	0.0027	0.1273
NE3	3.0017	0.6123	0.0017	0.6123	0.9895	0.6092	0.0031	0.1540

Table 15. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for log Weibull ($\mu = 0,\ \sigma = 3$) distributed data with 50,000 repetitions.

$n = 5$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
r = 0								
BLUE	2.9978	1.2009	0.0022	1.2009	1.0000			
NE1	2.9997	1.1719	0.0003	1.1719	1.0247	1.1682	0.0037	0.6288
NE2	2.9964	1.1341	0.0035	1.1341	1.0589	1.1347	0.0006	0.7021
$r = 1$								
BLUE	2.9991	1.8348	0.0009	1.8348	1.0000			
NE1	2.9939	1.7317	0.0061	1.7317	1.0595	1.7206	0.0111	1.7206
NE2	3.0074	1.6518	0.0074	1.6519	1.1108	1.6511	0.0007	1.6512

Table 16. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for log Weibull ($\mu = 0,\ \sigma = 3$) distributed data with 50,000 repetitions.

$n = 10$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	3.0056	0.7784	0.0056	0.7784	1.0000			
NE1	3.0024	0.8122	0.0024	0.8122	0.9584	0.8165	0.0043	0.2995
NE2	3.0048	0.7958	0.0048	0.7958	0.9781	0.8032	0.0074	0.3829
$r = 1$								
BLUE	3.0019	0.9319	0.0019	0.9319	1.0000			
NE1	3.0005	0.9305	0.0005	0.9305	1.0015	0.9299	0.0006	0.3829
NE2	2.9981	0.9088	0.0019	0.9088	1.0254	0.9063	0.0025	0.5014
$r = 2$								
BLUE	2.9984	1.1561	0.0016	1.1561	1.0000			
NE1	2.9948	1.1434	0.0052	1.1434	1.0111	1.1437	0.0003	0.5825
NE2	3.0020	1.1170	0.0020	1.1170	1.0350	1.1123	0.0046	0.6914
$r = 3$								
BLUE	3.0015	1.5112	0.0015	1.5112	1.0000			
NE1	2.9999	1.6037	0.0001	1.6037	0.9423	1.6057	0.0020	1.1517
NE2	3.0050	1.5526	0.0050	1.5526	0.9733	1.5495	0.0031	1.0685

Table 17. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for log Weibull ($\mu = 0,\ \sigma = 3$) distributed with 50,000 repetitions.

$n = 15$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	2.9993	0.6157	0.0007	0.6157	1.0000			
NE1	2.9933	0.6634	0.6634	0.6635	0.9281	0.6662	0.0028	0.2036
NE2	3.0033	0.6540	0.6540	0.6540	0.9414	0.6538	0.0002	0.2680
$r = 1$								
BLUE	3.0011	0.6936	0.0011	0.6936	1.0000			
NE1	2.9996	0.7162	0.0004	0.7162	0.9684	0.7183	0.0021	0.2322
NE2	3.0077	0.7063	0.0077	0.7064	0.9820	0.7046	0.0017	0.3185
$r = 2$								
BLUE	3.0001	0.7896	0.0001	0.7896	1.0000			
NE1	3.0053	0.7922	0.0053	0.7922	0.9967	0.7937	0.0015	0.2803
NE2	3.0038	0.7785	0.0038	0.7785	1.0143	0.7793	0.0008	0.3943
$r = 3$								
BLUE	2.9969	0.9077	0.0031	0.9077	1.0000			
NE1	3.0030	0.9108	0.0030	0.9108	0.9966	0.9100	0.0008	0.3656
NE2	2.9980	0.8926	0.0020	0.8926	1.0169	0.8905	0.0021	0.4914

Table 18. Expected values, standard deviations, absolute biases, rmse values an efficiencies associated with BLU and NE estimators for log Weibull ($\mu = 0,\ \sigma = 3$) distributed data with 50,000 repetitions.

$n = 20$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	2.9974	0.5234	0.0026	0.5234	1.0000			
NE1	2.9994	0.5764	0.0006	0.5763	0.9080	0.5792	0.0028	0.1582
NE2	3.0034	0.5678	0.0034	0.5678	0.9218	0.5674	0.0004	0.2095
$r = 1$								
BLUE	3.0022	0.5776	0.0022	0.5776	1.0000			
NE1	3.0026	0.6127	0.0026	0.6127	0.9427	0.6139	0.0012	0.1727
NE2	3.0009	0.6031	0.0009	0.6030	0.9577	0.6005	0.0026	0.2379
$r = 2$								
BLUE	2.9999	0.6350	0.0001	0.6350	1.0000			
NE1	2.9953	0.6550	0.0047	0.6550	0.9695	0.6532	0.0018	0.1936
NE2	2.9963	0.6447	0.0036	0.6447	0.9850	0.6450	0.0003	0.2801
$r = 3$								
BLUE	2.9998	0.6931	0.0002	0.6931	1.0000			
NE1	3.0002	0.7006	0.0002	0.7006	0.9893	0.7051	0.0045	0.2237
NE2	3.0010	0.6892	0.0010	0.6892	1.0057	0.6969	0.0078	0.3290

Table 19. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for normal ($\mu = 0,\ \sigma = 3$) distributed data with 50,000 repetitions.

$n = 5$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	2.9982	1.0970	0.0018	1.0970	1.0000			
NE1	3.0044	1.0079	0.0044	1.0079	1.0884	1.0062	0.0016	0.5090
NE2	3.0051	1.0327	0.0051	1.0327	1.0623	1.0246	0.0081	0.4843
NE3	2.9981	1.0025	0.0019	1.0025	1.0943	1.0054	0.0029	0.4802
$r = 1$								
BLUE	3.0000	1.7244	0.0000	1.7244	1.0000			
NE1	2.9968	1.3995	0.0032	1.3995	1.2322	1.3968	0.0028	0.9547
NE2	2.9978	1.4811	0.0022	1.4811	1.1643	1.4881	0.0070	1.1765
NE3	2.9941	1.3979	0.0059	1.3979	1.2336	1.3938	0.0041	0.9506

The power values in these tables are the ones that make p-values for the normality test largest. Therefore, the normality of relevant sampling distributions holds provided that the power values are chosen in some tight interval containing c values given in Tables 35–41. Hence, new test and confidence interval procedures involving the new estimators can be proposed for the scale parameter σ when populations have half normal, half logistic, exponential, Rayleigh, normal or log-Weibull distributions. The test statistics for the hypotheses about σ are defined as

$$Z = \frac{\left(\hat{\sigma}^k_{ne1}\right)^{c_1} - \sigma_0^{c_1}}{c_1 \sigma_0^{c_1 - 1} \sigma_{\hat{\sigma}^k_{ne1}}} \sim N(0,\ 1) \quad \text{or} \quad Z = \frac{\left(\hat{\sigma}^k_{ne1}\right)^{c_1} - \sigma_0^{c_1}}{c_1 \sigma_0^{c_1 - 1} S_{\hat{\sigma}^k_{ne1}}} \sim N(0,\ 1)$$

$$Z = \frac{\left(\hat{\sigma}^k_{ne2}\right)^{c_2} - \sigma_0^{c_2}}{c_2 \sigma_0^{c_2 - 1} \sigma_{\hat{\sigma}^k_{ne2}}} \sim N(0,\ 1) \quad \text{or} \quad Z = \frac{\left(\hat{\sigma}^k_{ne2}\right)^{c_2} - \sigma_0^{c_2}}{c_2 \sigma_0^{c_2 - 1} S_{\hat{\sigma}^k_{ne2}}} \sim N(0,\ 1)$$

$$Z = \frac{\left(\hat{\sigma}^k_{ne3}\right)^{c_3} - \sigma_0^{c_3}}{c_3 \sigma_0^{c_3 - 1} \sigma_{\hat{\sigma}^k_{ne3}}} \sim N(0,\ 1) \quad \text{or} \quad Z = \frac{\left(\hat{\sigma}^k_{ne3}\right)^{c_3} - \sigma_0^{c_3}}{c_3 \sigma_0^{c_3 - 1} S_{\hat{\sigma}^k_{ne3}}} \sim N(0,\ 1)$$

Table 20. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for normal ($\mu = 0,\ \sigma = 3$) distributed data with 50,000 repetitions.

$n = 10$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	2.9967	0.7177	0.0033	0.7177	1.0000			
NE1	2.9953	0.6977	0.0047	0.6978	1.0287	0.7044	0.0067	0.2674
NE2	2.9942	0.7089	0.0058	0.7089	1.0124	0.7135	0.0046	0.2213
NE3	2.9923	0.6957	0.0077	0.6957	1.0316	0.7009	0.0052	0.2421
$r = 1$								
BLUE	2.9962	0.8593	0.0038	0.8593	1.0000			
NE1	2.9985	0.8198	0.0015	0.8198	1.0482	0.8190	0.0008	0.4173
NE2	3.0002	0.8372	0.0002	0.8372	1.0264	0.8334	0.0038	0.2995
NE3	2.9953	0.8168	0.0047	0.8168	1.0520	0.8162	0.0006	0.3880
$r = 2$								
BLUE	2.9978	1.0807	0.0022	1.0807	1.0000			
NE1	3.0063	0.9926	0.0063	0.9926	1.0888	0.9829	0.0097	0.5945
NE2	3.0070	1.0202	0.0070	1.0202	1.0593	1.0157	0.0044	0.4409
NE3	2.9999	0.9891	0.0001	0.9891	1.0926	0.9912	0.0021	0.5883
$r = 3$								
BLUE	2.9908	1.4842	0.0092	1.4842	1.0000			
NE1	3.0053	1.2941	0.0053	1.2941	1.1469	1.2882	0.0058	0.8366
NE2	3.0080	1.3469	0.0080	1.3469	1.1019	1.3491	0.0022	0.8384
NE3	2.9972	1.2904	0.0028	1.2904	1.1502	1.2947	0.0043	0.8399

Table 21. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for normal ($\mu = 0,\ \sigma = 3$) distributed with 50,000 repetitions.

$n = 15$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE($\hat{\sigma}$)	EFF($\hat{\sigma}$)	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
r = 0								
BLUE	3.0014	0.5749	0.0014	0.5749	1.0000			
NE1	2.9973	0.5662	0.0027	0.5662	1.0154	0.5620	0.0043	0.1832
NE2	2.9969	0.5761	0.0031	0.5761	0.9979	0.5728	0.0033	0.1490
NE3	3.0000	0.5677	0.0000	0.5677	1.0127	0.5679	0.0002	0.1684
$r = 1$								
BLUE	3.0000	0.6441	0.0000	0.6441	1.0000			
NE1	2.9977	0.6314	0.0023	0.6314	1.0201	0.6270	0.0044	0.2567
NE2	2.9986	0.6420	0.0014	0.6420	1.0033	0.6386	0.0034	0.1795
NE3	3.0036	0.6316	0.0036	0.6316	1.0198	0.6275	0.0041	0.2295
$r = 2$								
BLUE	2.9994	0.7313	0.0006	0.7313	1.0000			
NE1	3.0022	0.7117	0.0022	0.7117	1.0275	0.7055	0.0062	0.3566
NE2	3.0038	0.7247	0.0038	0.7247	1.0091	0.7202	0.0046	0.2237
NE3	3.0020	0.7101	0.0020	0.7101	1.0299	0.7078	0.0023	0.3317
$r = 3$								
BLUE	2.9951	0.8458	0.0049	0.8458	1.0000			
NE1	3.0006	0.8073	0.0006	0.8073	1.0477	0.8158	0.0085	0.4783
NE2	3.0001	0.8243	0.0001	0.8243	1.0261	0.8283	0.0040	0.2892
NE3	2.9950	0.8043	0.0050	0.8043	1.0516	0.8101	0.0058	0.4591

And the confidence intervals for σ with $100(1-\alpha)\%$ confidence level using $\hat{\sigma}^k_{ne1}$, $\hat{\sigma}^k_{ne2}$ and, $\hat{\sigma}^k_{ne3}$ are

$$\left(\left[\left(\hat{\sigma}^k_{ne1}\right)^{c_1} - z_{\alpha/2}c_1\left(\hat{\sigma}^k_{ne1}\right)^{c_1-1}\sigma_{\hat{\sigma}^k_{ne1}}\right]^{1/c_1}, \left[\left(\hat{\sigma}^k_{ne1}\right)^{c_1} + z_{\alpha/2}c_1\left(\hat{\sigma}^k_{ne1}\right)^{c_1-1}\sigma_{\hat{\sigma}^k_{ne1}}\right]^{1/c_1}\right) \text{ or}$$

$$\left(\left[\left(\hat{\sigma}^k_{ne1}\right)^{c_1} - z_{\alpha/2}c_1\left(\hat{\sigma}^k_{ne1}\right)^{c_1-1}S_{\hat{\sigma}^k_{ne1}}\right]^{1/c_1}, \left[\left(\hat{\sigma}^k_{ne1}\right)^{c_1} + z_{\alpha/2}c_1\left(\hat{\sigma}^k_{ne1}\right)^{c_1-1}S_{\hat{\sigma}^k_{ne1}}\right]^{1/c_1}\right)$$

Table 22. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for normal ($\mu = 0,\ \sigma = 3$) distributed data with 50,000 repetitions.

$n = 20$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE($\hat{\sigma}$)	EFF($\hat{\sigma}$)	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	3.0012	0.4922	0.0012	0.4922	1.0000			
NE1	3.0021	0.4885	0.0021	0.4885	1.0076	0.4884	0.0001	0.1421
NE2	3.0022	0.4983	0.0022	0.4983	0.9878	0.4964	0.0019	0.1156
NE3	3.0025	0.4901	0.0025	0.4901	1.0043	0.4911	0.0010	0.1310
$r = 1$								
BLUE	3.0005	0.5328	0.0005	0.5328	1.0000			
NE1	2.9988	0.5267	0.0012	0.5267	1.0116	0.5332	0.0065	0.1859
NE2	2.9975	0.5352	0.0024	0.5352	0.9955	0.5348	0.0004	0.1311
NE3	3.0009	0.5266	0.0009	0.5266	1.0118	0.5290	0.0024	0.1634
$r = 2$								
BLUE	3.0002	0.5828	0.0002	0.5828	1.0000			
NE1	3.0024	0.5753	0.0024	0.5753	1.0130	0.5776	0.0023	0.2437
NE2	3.0024	0.5840	0.0024	0.5840	0.9979	0.5893	0.0053	0.1550
NE3	2.9973	0.5735	0.0027	0.5735	1.0162	0.5748	0.0013	0.2166
$r = 3$								
BLUE	3.0005	0.6470	0.0005	0.6470	1.0000			
NE1	2.9987	0.6342	0.0013	0.6342	1.0202	0.6336	0.0006	0.3155
NE2	2.9982	0.6466	0.0018	0.6466	1.0006	0.6457	0.0008	0.1822
NE3	2.9989	0.6332	0.0010	0.6332	1.0218	0.6287	0.0045	0.2892

Table 23. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for exponential ($\sigma = 3$) distributed data with 50,000 repetitions.

$n=5$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r=0$								
BLUE	2.9930	1.3335	0.0070	1.3336	1.0000			
NE1	2.9942	1.3902	0.0058	1.3902	0.9592	1.3924	0.0022	0.9368
NE2	2.9960	1.3359	0.0040	1.3359	0.9982	1.3376	0.0017	0.6087
NE3	2.9962	1.3423	0.0038	1.3423	0.9934	1.3438	0.0015	0.8048
$r=1$								
BLUE	2.9962	1.5025	0.0038	1.5025	1.0000			
NE1	2.9909	1.5107	0.0091	1.5107	0.9946	1.5170	0.0063	1.1177
NE2	2.9931	1.5066	0.0068	1.5066	0.9973	1.5136	0.0070	0.7920
NE3	2.9904	1.4999	0.0096	1.4999	1.0017	1.5000	0.0001	1.0896

$$\left(\left[\left(\hat{\sigma}^k_{ne2}\right)^{c_2} - z_{\alpha/2}c_2\left(\hat{\sigma}^k_{ne2}\right)^{c_2-1}\sigma_{\hat{\sigma}^k_{ne2}}\right]^{1/c_2}, \left[\left(\hat{\sigma}^k_{ne2}\right)^{c_2} + z_{\alpha/2}c_2\left(\hat{\sigma}^k_{ne2}\right)^{c_2-1}\sigma_{\hat{\sigma}^k_{ne2}}\right]^{1/c_2}\right) \text{ or}$$

$$\left(\left[\left(\hat{\sigma}^k_{ne2}\right)^{c_2} - z_{\alpha/2}c_2\left(\hat{\sigma}^k_{ne2}\right)^{c_2-1}S_{\hat{\sigma}^k_{ne2}}\right]^{1/c_2}, \left[\left(\hat{\sigma}^k_{ne2}\right)^{c_2} + z_{\alpha/2}c_2\left(\hat{\sigma}^k_{ne2}\right)^{c_2-1}S_{\hat{\sigma}^k_{ne2}}\right]^{1/c_2}\right)$$

$$\left(\left[\left(\hat{\sigma}^k_{ne3}\right)^{c_3} - z_{\alpha/2}c_3\left(\hat{\sigma}^k_{ne3}\right)^{c_3-1}\sigma_{\hat{\sigma}^k_{ne3}}\right]^{1/c_3}, \left[\left(\hat{\sigma}^k_{ne3}\right)^{c_3} + z_{\alpha/2}c_3\left(\hat{\sigma}^k_{ne3}\right)^{c_3-1}\sigma_{\hat{\sigma}^k_{ne3}}\right]^{1/c_3}\right) \text{ or}$$

$$\left(\left[\left(\hat{\sigma}^k_{ne3}\right)^{c_3} - z_{\alpha/2}c_3\left(\hat{\sigma}^k_{ne3}\right)^{c_3-1}S_{\hat{\sigma}^k_{ne3}}\right]^{1/c_3}, \left[\left(\hat{\sigma}^k_{ne3}\right)^{c_3} + z_{\alpha/2}c_3\left(\hat{\sigma}^k_{ne3}\right)^{c_3-1}S_{\hat{\sigma}^k_{ne3}}\right]^{1/c_3}\right)$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of standard normal distribution.

Table 24. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for exponential ($\sigma = 3$) distributed data with 50,000 repetitions.

$n=10$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r=0$								
BLUE	2.9995	0.9533	0.0005	0.9533	1.0000			
NE1	2.9959	1.0059	0.0041	1.0059	0.9477	1.0000	0.0059	0.5106
NE2	2.9977	0.9536	0.0023	0.9536	0.9997	0.9475	0.0061	0.3045
NE3	2.9985	0.9595	0.0014	0.9595	0.9935	0.9520	0.0075	0.4026
$r=1$								
BLUE	2.9988	0.9986	0.0012	0.9986	1.0000			
NE1	3.0016	1.0280	0.0016	1.0279	0.9714	1.0318	0.0038	0.4866
NE2	3.0003	1.0004	0.0003	1.0004	0.9982	1.0035	0.0031	0.3382
NE3	3.0043	1.0041	0.0043	1.0041	0.9945	1.0068	0.0027	0.4344
$r=2$								
BLUE	3.0020	1.0545	0.0020	1.0545	1.0000			
NE1	3.0050	1.0699	0.0050	1.0700	0.9856	1.0743	0.0043	0.5239
NE2	3.0045	1.0581	0.0045	1.0581	0.9966	1.0633	0.0052	0.3785
NE3	3.0085	1.0579	0.0085	1.0579	0.9968	1.0613	0.0034	0.5016
$r=3$								
BLUE	2.9969	1.1385	0.0031	1.1385	1.0000			
NE1	3.0021	1.1424	0.0021	1.1424	0.9966	1.1439	0.0015	0.6775
NE2	3.0033	1.1469	0.0033	1.1469	0.9927	1.1488	0.0019	0.4440
NE3	2.9916	1.1378	0.0084	1.1378	1.0006	1.1383	0.0005	0.6706

Table 25. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for exponential ($\sigma = 3$) distributed with 50,000 repetitions.

$n = 15$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	3.0017	0.7741	0.0017	0.7741	1.0000			
NE1	3.0028	0.8272	0.0028	0.8272	0.9358	0.8269	0.0003	0.3587
NE2	3.0034	0.7749	0.0034	0.7749	0.9990	0.7730	0.0019	0.2012
NE3	3.0016	0.7807	0.0016	0.7807	0.9915	0.7799	0.0008	0.2696
$r = 1$								
BLUE	3.0027	0.8012	0.0027	0.8012	1.0000			
NE1	3.0088	0.8356	0.0088	0.8357	0.9588	0.8314	0.0042	0.3301
NE2	3.0089	0.8034	0.0089	0.8035	0.9973	0.8039	0.0004	0.2166
NE3	3.0065	0.8070	0.0065	0.8070	0.9928	0.8069	0.0001	0.2818
$r = 2$								
BLUE	3.0030	0.8330	0.0030	0.8330	1.0000			
NE1	3.0079	0.8534	0.0079	0.8534	0.9761	0.8548	0.0014	0.3272
NE2	2.9969	0.8324	0.0031	0.8324	1.0007	0.8346	0.0022	0.2340
NE3	2.9997	0.8339	0.0003	0.8339	0.9989	0.8367	0.0028	0.2972
$r = 3$								
BLUE	3.0037	0.8693	0.0037	0.8693	1.0000			
NE1	3.0096	0.8846	0.0096	0.8847	0.9827	0.8838	0.0008	0.3492
NE2	3.0075	0.8722	0.0075	0.8722	0.9967	0.8694	0.0028	0.2542
NE3	3.0058	0.8711	0.0058	0.8712	0.9979	0.8689	0.0022	0.3303

4. Real data example

Let us consider the data given by Proschan (1963). The data are arranged in increasing order of magnitude and show times between successive failures of air conditioning equipment in a Boeing 720 airplane (Balakrishnan and Cohen, 1991).

$$12, 21, 26, 27, 29, 29, 48, 57, 59, 70, 74, 153, 326, 386, 502$$

Table 26. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for exponential ($\sigma = 3$) distributed data with 50,000 repetitions.

$n = 20$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	3.0050	0.6686	0.0050	0.6686	1.0000			
NE1	2.9976	0.7167	0.0024	0.7167	0.9329	0.7171	0.0004	0.2781
NE2	2.9997	0.6677	0.0003	0.6677	1.0013	0.6703	0.0026	0.1506
NE3	3.0046	0.6742	0.0046	0.6742	0.9917	0.6810	0.0068	0.2037
$r = 1$								
BLUE	3.0043	0.6867	0.0043	0.6867	1.0000			
NE1	3.0077	0.7188	0.0077	0.7189	0.9553	0.7223	0.0035	0.2522
NE2	3.0020	0.6867	0.0020	0.6867	1.0000	0.6889	0.0021	0.1590
NE3	2.9989	0.6897	0.0011	0.6897	0.9957	0.6924	0.0027	0.2081
$r = 2$								
BLUE	3.0025	0.7123	0.0025	0.7123	1.0000			
NE1	3.0003	0.7358	0.0003	0.7358	0.9681	0.7288	0.0070	0.2480
NE2	3.0000	0.7123	0.0000	0.7123	1.0000	0.7057	0.0066	0.1691
NE3	3.0004	0.7154	0.0004	0.7154	0.9957	0.7090	0.0063	0.2171
$r = 3$								
BLUE	3.0021	0.7240	0.0021	0.7240	1.0000			
NE1	3.0061	0.7412	0.0061	0.7412	0.9768	0.7483	0.0071	0.2479
NE2	3.0046	0.7256	0.0046	0.7256	0.9978	0.7329	0.0073	0.1786
NE3	3.0051	0.7265	0.0051	0.7265	0.9966	0.7313	0.0048	0.2258

Table 27. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for Rayleigh ($\sigma = 3$) distributed data with 50,000 repetitions.

$n = 5$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	3.0001	0.6809	0.0001	0.6809	1.0000			
NE1	2.9963	0.6803	0.0037	0.6803	1.0009	0.6849	0.0046	0.2743
NE2	2.9967	0.6811	0.0033	0.6811	0.9997	0.6779	0.0032	0.1582
NE3	3.0005	0.6815	0.0005	0.6815	0.9991	0.6797	0.0018	0.2741
$r = 1$								
BLUE	3.0004	0.7632	0.0004	0.7632	1.0000			
NE1	3.0048	0.7669	0.0048	0.7669	0.9952	0.7714	0.0045	0.4502
NE2	2.9993	0.7641	0.0007	0.7641	0.9988	0.7658	0.0017	0.1982
NE3	3.0011	0.7642	0.0011	0.7642	0.9987	0.7620	0.0022	0.4449

The data are assumed to come from in turn exponential distribution, Rayleigh distribution, and half logistic distributions. Tables 42–44 present the results of the estimation associated with the methods BLU and NE's for the cases of complete data ($r = 0$) and symmetrically doubly Type II censored data with the degree of $r = 1, 2, 3$ for the exponential, Rayleigh, and half logistic distributions respectively. The value of k was taken as 1,000 in the computation of $\hat{\sigma}^k_{ne1}$, $\hat{\sigma}^k_{ne2}$ and, $\hat{\sigma}^k_{ne3}$ and their standard error estimates.

It is noticeable that the results for the methods BLU and NE2 in general agree with each other in comparison to NE1 and NE3 for all of the three distributions. In addition, the estimates associated with the NE1, NE2 and NE3 tend to be close to BLU in the case of Rayleigh distribution. Further, the estimates of standard error for NE2 are smaller than those of NE1 and NE3. Furthermore, 95% confidence interval estimates for the parameter σ are obtained as defined in Section 3 since the sampling distributions of some power of $\hat{\sigma}^k_{ne1}$, $\hat{\sigma}^k_{ne2}$ and, $\hat{\sigma}^k_{ne3}$ have been shown empirically to be normally distributed. The results are again displayed in Tables 42–44.

Table 28. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for Rayleigh ($\sigma = 3$) distributed data with 50,000 repetitions.

$n = 10$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	3.0007	0.4771	0.0007	0.4771	1.0000			
NE1	3.0015	0.4775	0.0015	0.4774	0.9992	0.4852	0.0077	0.1332
NE2	2.9998	0.4778	0.0002	0.4778	0.9985	0.4773	0.0005	0.0788
NE3	3.0039	0.4783	0.0039	0.4783	0.9975	0.4768	0.0015	0.1305
$r = 1$								
BLUE	3.0003	0.5019	0.0003	0.5019	1.0000			
NE1	3.0015	0.5023	0.0015	0.5023	0.9992	0.5073	0.0050	0.1528
NE2	3.0010	0.5024	0.0010	0.5024	0.9990	0.5042	0.0018	0.0869
NE3	3.0011	0.5022	0.0011	0.5022	0.9994	0.5056	0.0034	0.1530
$r = 2$								
BLUE	2.9983	0.5363	0.0017	0.5363	1.0000			
NE1	2.9968	0.5372	0.0032	0.5372	0.9983	0.5409	0.0037	0.1989
NE2	2.9988	0.5368	0.0012	0.5368	0.9991	0.5346	0.0022	0.0976
NE3	2.9959	0.5360	0.0041	0.5361	1.0006	0.5319	0.0041	0.1956
$r = 3$								
BLUE	3.0027	0.5737	0.0027	0.5737	1.0000			
NE1	3.0057	0.5777	0.0057	0.5777	0.9931	0.5820	0.0043	0.2904
NE2	3.0021	0.5757	0.0021	0.5757	0.9965	0.5782	0.0025	0.1123
NE3	3.0001	0.5750	0.0001	0.5750	0.9977	0.5758	0.0008	0.2873

Table 29. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for Rayleigh ($\sigma = 3$) distributed with 50,000 repetitions.

$n=15$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE($\hat{\sigma}$)	EFF($\hat{\sigma}$)	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r=0$								
BLUE	2.9979	0.3884	0.0021	0.3885	1.0000			
NE1	3.0006	0.3889	0.0006	0.3889	0.9987	0.3926	0.0037	0.0880
NE2	2.9973	0.3890	0.0027	0.3890	0.9985	0.3896	0.0005	0.0530
NE3	2.9982	0.3890	0.0018	0.3890	0.9985	0.3899	0.0009	0.0865
$r=1$								
BLUE	3.0003	0.4024	0.0003	0.4024	1.0000			
NE1	3.0042	0.4030	0.0042	0.4030	0.9985	0.4097	0.0067	0.0954
NE2	2.9999	0.4028	0.0001	0.4028	0.9990	0.4047	0.0018	0.0565
NE3	3.0023	0.4029	0.0023	0.4029	0.9988	0.4011	0.0018	0.0940
ir $=2$								
BLUE	2.9981	0.4183	0.0019	0.4183	1.0000			
NE1	3.0014	0.4190	0.0014	0.4190	0.9983	0.4244	0.0054	0.1082
NE2	2.9975	0.4185	0.0025	0.4185	0.9995	0.4191	0.0006	0.0604
NE3	3.0000	0.4187	0.0000	0.4187	0.9990	0.4205	0.0018	0.1074
$r=3$								
BLUE	3.0012	0.4347	0.0012	0.4347	1.0000			
NE1	3.0052	0.4353	0.0005	0.4353	0.9986	0.4410	0.0057	0.1286
NE2	3.0043	0.4354	0.0043	0.4354	0.9984	0.4375	0.0021	0.0650
NE3	3.0030	0.4350	0.0030	0.4350	0.9993	0.4352	0.0002	0.1269

Table 30. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for Rayleigh ($\sigma = 3$) distributed data with 50,000 repetitions.

$n=20$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE($\hat{\sigma}$)	EFF($\hat{\sigma}$)	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r=0$								
BLUE	3.0019	0.3369	0.0019	0.3369	1.0000			
NE1	3.0000	0.3368	0.0000	0.3368	1.0003	0.3393	0.0025	0.0658
NE2	3.0007	0.3375	0.0007	0.3375	0.9982	0.3368	0.0007	0.0398
NE3	3.0018	0.3376	0.0018	0.3376	0.9979	0.3378	0.0002	0.0647
$r=1$								
BLUE	2.9986	0.3441	0.0014	0.3441	1.0000			
NE1	2.9986	0.3441	0.0014	0.3441	1.0000	0.3486	0.0045	0.0687
NE2	2.9975	0.3443	0.0025	0.3443	0.9994	0.3454	0.0011	0.0416
NE3	2.9998	0.3444	0.0002	0.3444	0.9991	0.3474	0.0030	0.0687
$r=2$								
BLUE	2.9990	0.3567	0.0010	0.3567	1.0000			
NE1	2.9982	0.3568	0.0018	0.3568	0.9997	0.3588	0.0020	0.0746
NE2	2.9983	0.3569	0.0017	0.3569	0.9994	0.3531	0.0038	0.0438
NE3	2.9996	0.3569	0.0004	0.3569	0.9994	0.3558	0.0011	0.0743
$r=3$								
BLUE	3.0025	0.3646	0.0025	0.3647	1.0000			
NE1	3.0032	0.3650	0.0032	0.3650	0.9989	0.3705	0.0055	0.0832
NE2	3.0023	0.3649	0.0023	0.3649	0.9992	0.3656	0.0007	0.0461
NE3	3.0034	0.3649	0.0034	0.3649	0.9992	0.3655	0.0006	0.0822

Table 31. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for uniform (0, $\sigma = 3$) distributed data with 50,000 repetitions.

$n=5$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE($\hat{\sigma}$)	EFF($\hat{\sigma}$)	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r=0$								
BLUE	2.9998	0.5106	0.0002	0.5106	1.0000			
NE1	3.0031	0.5322	0.0031	0.5323	0.9594	0.5300	0.0022	0.2472
NE2	2.9977	0.5113	0.0023	0.5113	0.9986	0.5096	0.0017	0.4408
$r=1$								
BLUE	2.9986	0.8048	0.0014	0.8048	1.0000			
NE1	3.0030	0.8427	0.0030	0.8427	0.9550	0.8424	0.0003	0.4626
NE2	3.0010	0.8055	0.0010	0.8055	0.9991	0.8069	0.0014	0.7583

Table 32. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for uniform (0, $\sigma = 3$) distributed data with 50,000 repetitions.

$n = 10$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	3.0000	0.2749	0.0000	0.2749	1.0000			
NE1	2.9980	0.2837	0.0020	0.2837	0.9690	0.2799	0.0038	0.0908
NE2	3.0004	0.2772	0.0004	0.2772	0.9917	0.2709	0.0063	0.2115
$r = 1$								
BLUE	3.0015	0.4093	0.0015	0.4093	1.0000			
NE1	3.0057	0.4208	0.0057	0.4209	0.9727	0.4198	0.0010	0.1317
NE2	3.0023	0.4098	0.0023	0.4098	0.9988	0.4108	0.0010	0.3378
$r = 2$								
BLUE	2.9982	0.5286	0.0018	0.5286	1.0000			
NE1	2.9970	0.5423	0.0030	0.5423	0.9747	0.5427	0.0004	0.1810
NE2	2.9986	0.5289	0.0014	0.5289	0.9994	0.5297	0.0008	0.4453
$r = 3$								
BLUE	3.0010	0.6549	0.0010	0.6549	1.0000			
NE1	3.0021	0.6738	0.0021	0.6738	0.9720	0.6704	0.0034	0.2501
NE2	3.0022	0.6553	0.0022	0.6553	0.9994	0.6573	0.0020	0.5854

Table 33. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for uniform (0, $\sigma = 3$) distributed with 50,000 repetitions.

$n = 15$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
$r = 0$								
BLUE	3.0015	0.1867	0.0015	0.1867	1.0000			
NE1	3.0016	0.1939	0.0016	0.1939	0.9629	0.1940	0.0001	0.0557
NE2	3.0015	0.1896	0.0015	0.1896	0.9847	0.1911	0.0015	0.1384
$r = 1$								
BLUE	2.9996	0.2747	0.0004	0.2747	1.0000			
NE1	2.9999	0.2816	0.0001	0.2816	0.9755	0.2807	0.0009	0.0752
NE2	2.9986	0.2754	0.0014	0.2754	0.9975	0.2748	0.0006	0.2045
$r = 2$								
BLUE	2.9987	0.3498	0.0013	0.3498	1.0000			
NE1	2.9959	0.3576	0.0041	0.3576	0.9782	0.3581	0.0005	0.0943
NE2	2.9999	0.3503	0.0001	0.3503	0.9986	0.3502	0.0001	0.2718
$r = 3$								
BLUE	3.0012	0.4206	0.0012	0.4206	1.0000			
NE1	3.0012	0.4299	0.0012	0.4299	0.9784	0.4299	0.0000	0.1172
NE2	3.0024	0.4210	0.0024	0.4210	0.9990	0.4155	0.0055	0.3272

Table 34. Expected values, standard deviations, absolute biases, rmse values and efficiencies associated with BLU and NE estimators for uniform (0, $\sigma = 3$) distributed data with 50,000 repetitions.

$n = 20$	$E(\hat{\sigma})$	$\sigma_{\hat{\sigma}}$	$\lvert Bias(\hat{\sigma})\rvert$	RMSE$(\hat{\sigma})$	EFF$(\hat{\sigma})$	$E(S_{\hat{\sigma}^k_{ne}})$	$\lvert Bias(S_{\hat{\sigma}^k_{ne}})\rvert$	$\sigma(S_{\hat{\sigma}^k_{ne}})$
r = 0								
BLUE	3.0001	0.1430	0.0001	0.1430	1.0000			
NE1	2.9995	0.1494	0.0005	0.1494	0.9572	0.1493	0.0001	0.0409
NE2	3.0003	0.1467	0.0003	0.1467	0.9748	0.1483	0.0016	0.1003
$r = 1$								
BLUE	2.9999	0.2074	0.0001	0.2074	1.0000			
NE1	2.9994	0.2126	0.0006	0.2126	0.9755	0.2135	0.0009	0.0517
NE2	3.0002	0.2086	0.0002	0.2086	0.9942	0.2074	0.0011	0.1446
$r = 2$								
BLUE	2.9990	0.2599	0.0010	0.2599	1.0000			
NE1	2.9992	0.2656	0.0008	0.2656	0.9785	0.2658	0.0002	0.0627
NE2	2.9997	0.2604	0.0003	0.2604	0.9981	0.2630	0.0026	0.1864
$r = 3$								
BLUE	2.9977	0.3097	0.0023	0.3097	1.0000			
NE1	2.9956	0.3160	0.0043	0.3161	0.9801	0.3194	0.0034	0.0757
NE2	3.0000	0.3101	0.0000	0.3101	0.9987	0.3115	0.0014	0.2267

Table 35. Goodness of fit tests for normality of sampling distributions for some power of $\hat{\sigma}^k_{ne1}$, $\hat{\sigma}^k_{ne2}$ and $\hat{\sigma}^k_{ne3}$ in the case of half normal distribution via Anderson-Darling tests.

		$(\hat{\sigma}^k_{ne1})^{c_1}$			$(\hat{\sigma}^k_{ne2})^{c_2}$			$(\hat{\sigma}^k_{ne3})^{c_3}$		
		c_1	A-D	p-value	c_2	A-D	p-value	c_3	A-D	P-value
$n=5$	$r=0$	0.61	0.488	0.223	0.55	0.252	0.740	0.59	1.658	<0.005*
	$r=1$	0.58	1.887	<0.005*	0.51	0.466	0.253	0.58	2.206	<0.005*
	$r=0$	0.63	0.387	0.388	0.58	0.293	0.604	0.66	0.476	0.238
	$r=1$	0.62	0.470	0.247	0.58	0.458	0.264	0.63	0.514	0.193
$n=10$	$r=2$	0.60	0.425	0.317	0.55	0.320	0.532	0.61	0.757	0.049
	$r=3$	0.55	0.513	0.194	0.51	0.432	0.304	0.58	0.619	0.107
	$r=0$	0.68	0.458	0.264	0.61	0.394	0.374	0.66	0.350	0.474
	$r=1$	0.66	0.404	0.355	0.61	0.374	0.416	0.61	0.233	0.799
$n=15$	$r=2$	0.64	0.194	0.893	0.59	0.220	0.836	0.64	0.281	0.643
	$r=3$	0.62	0.148	0.965	0.56	0.239	0.779	0.60	1.455	<0.005*
	$r=0$	0.64	0.269	0.680	0.61	0.255	0.727	0.62	0.395	0.373
$n=20$	$r=1$	0.67	0.218	0.841	0.62	0.132	0.982	0.59	0.541	0.165
	$r=2$	0.65	0.333	0.510	0.60	0.286	0.626	0.66	0.224	0.823
	$r=3$	0.62	0.340	0.502	0.58	0.472	0.244	0.62	0.547	0.160

* Ryan-Joiner test statistics $= 1.000$, p-value > 0.098 and Kolmogorov-Smirnov test statistics $= 0.005$ or 0.006, p-value < 0.010.

Table 36. Goodness of fit tests for normality of sampling distributions for some power of $\hat{\sigma}^k_{ne1}$, $\hat{\sigma}^k_{ne2}$ and $\hat{\sigma}^k_{ne3}$ in the case of half logistic distribution via Anderson-Darling tests.

		$(\hat{\sigma}^k_{ne1})^{c_1}$			$(\hat{\sigma}^k_{ne2})^{c_2}$			$(\hat{\sigma}^k_{ne3})^{c_3}$		
		c_1	A-D	p-value	c_2	A-D	P-value	c_3	A-D	p-value
$n=5$	$r=0$	0.33	0.427	0.313	0.35	0.178	0.919	0.39	0.376	0.411
	$r=1$	0.43	0.307	0.564	0.38	0.359	0.451	0.42	0.334	0.509
	$r=0$	0.34	0.745	0.052	0.40	0.296	0.594	0.40	0.229	0.809
	$r=1$	0.40	0.318	0.538	0.36	0.372	0.421	0.42	0.236	0.790
$n=10$	$r=2$	0.42	0.176	0.924	0.37	0.201	0.882	0.42	0.242	0.770
	$r=3$	0.43	0.356	0.458	0.39	0.513	0.194	0.45	0.173	0.928
	$r=0$	0.33	0.491	0.220	0.39	0.364	0.439	0.38	0.274	0.664
	$r=1$	0.31	0.269	0.681	0.34	0.382	0.399	0.39	0.194	0.894
$n=15$	$r=2$	0.43	0.280	0.646	0.41	0.282	0.638	0.43	0.230	0.806
	$r=3$	0.42	0.299	0.585	0.40	0.186	0.906	0.43	0.179	0.918
	$r=0$	0.25	0.190	0.900	0.34	0.419	0.327	0.39	0.183	0.911
$n=20$	$r=1$	0.31	0.687	0.073	0.37	0.455	0.269	0.39	0.369	0.428
	$r=2$	0.38	0.628	0.102	0.39	0.344	0.488	0.38	0.489	0.222
	$r=3$	0.40	0.239	0.781	0.40	0.186	0.907	0.43	0.344	0.487

Table 37. Goodness of fit tests for normality of sampling distributions for some power of $\hat{\sigma}^k_{ne1}$, $\hat{\sigma}^k_{ne2}$ and $\hat{\sigma}^k_{ne3}$ in the case of normal distribution via Anderson-Darling tests.

		$(\hat{\sigma}^k_{ne1})^{c}$			$(\hat{\sigma}^k_{ne2})^{c_2}$			$(\hat{\sigma}^k_{ne3})^{c_3}$		
		c_1	A-D	p-value	c_2	A-D	p-value	c_3	A-D	p-Value
$n=5$	$r=0$	0.58	1.233	<0.005*	0.54	0.712	0.063	0.59	1.266	<0.005*
	$r=1$	0.50	1.510	<0.005*	0.41	0.632	0.099	0.49	1.730	<0.005*
	$r=0$	0.60	0.489	0.222	0.55	0.475	0.240	0.59	0.393	0.376
	$r=1$	0.55	0.296	0.594	0.52	0.201	0.881	0.57	1.204	<0.005*
$n=10$	$r=2$	0.51	0.638	0.096	0.46	0.462	0.259	0.54	1.237	<0.005*
	$r=3$	0.44	0.816	0.035	0.39	0.281	0.642	0.43	0.527	0.179
	$r=0$	0.66	0.531	0.175	0.60	0.306	0.567	0.60	0.185	0.907
	$r=1$	0.58	0.522	0.184	0.54	0.288	0.618	0.61	0.695	0.069
$n=15$	$r=2$	0.56	0.579	0.133	0.52	0.333	0.510	0.56	0.327	0.519
	$r=3$	0.54	0.478	0.236	0.51	0.493	0.217	0.53	0.438	0.295
	$r=0$	0.65	0.721	0.060	0.61	0.468	0.249	0.59	0.435	0.300
$n=20$	$r=1$	0.66	0.200	0.883	0.64	0.275	0.662	0.58	0.258	0.717
	$r=2$	0.57	0.136	0.978	0.53	0.232	0.801	0.60	0.490	0.221
	$r=3$	0.54	0.292	0.605	0.51	0.139	0.975	0.56	0.449	0.278

* Ryan-Joiner test statistics $= 1.000$, p-value > 0.1000 and Kolmogorov-Smirnov test statistics $= 0.004$, $0.037 \leq p$-value $\leq$. 0.051

Table 38. Goodness of fit tests for normality of sampling distributions for some power of $\hat{\sigma}^k_{ne1}$, $\hat{\sigma}^k_{ne2}$ and $\hat{\sigma}^k_{ne3}$ in the case of exponential distribution via Anderson-Darling tests.

		$(\hat{\sigma}^k_{ne1})^{c_1}$			$(\hat{\sigma}^k_{ne2})^{c}$			$(\hat{\sigma}^k_{ne3})^{c_3}$		
		c_1	A-D	*p*-value	c_2	A-D	*p*-value	c_3	A-D	*p*-value
$n=5$	$r=0$	0.28	0.494	0.216	0.28	0.286	0.626	0.33	0.347	0.481
	r = 1	0.31	0.732	0.057	0.29	0.556	0.151	0.33	0.691	0.071
	$r=0$	0.25	0.294	0.601	0.28	0.267	0.687	0.33	0.422	0.321
	$r=1$	0.29	0.286	0.625	0.30	0.337	0.504	0.33	0.248	0.752
$n=10$	$r=2$	0.31	0.250	0.746	0.29	0.268	0.684	0.33	0.442	0.289
	$r=3$	0.31	0.271	0.676	0.28	0.365	0.437	0.31	0.415	0.335
	$r=0$	0.24	0.631	0.100	0.31	0.232	0.801	0.33	0.180	0.917
	$r=1$	0.28	0.226	0.818	0.30	0.212	0.856	0.32	0.448	0.279
$n=15$	$r=2$	0.31	0.456	0.267	0.31	0.256	0.723	0.34	0.188	0.902
	$r=3$	0.31	0.165	0.942	0.30	0.343	0.491	0.32	0.460	0.261
	$r=0$	0.23	0.476	0.238	0.29	0.272	0.671	0.33	0.181	0.914
$n=20$	$r=1$	0.28	0.162	0.946	0.31	0.117	0.990	0.29	0.252	0.738
	$r=2$	0.30	0.164	0.943	0.31	0.192	0.895	0.33	0.388	0.388
	$r=3$	0.31	0.165	0.942	0.28	0.250	0.744	0.34	0.474	0.242

Table 39. Goodness of fit tests for normality of sampling distributions for some power of $\hat{\sigma}^k_{ne1}$, $\hat{\sigma}^k_{ne2}$ and $\hat{\sigma}^k_{ne3}$ in the case of Rayleigh distribution via Anderson-Darling tests.

		$(\hat{\sigma}^k_{ne1})^{c_1}$			$(\hat{\sigma}^k_{ne2})^{c_2}$			$(\hat{\sigma}^k_{ne3})^{c_3}$		
		c_1	A-D	P-value	c_2	A-D	*p*-value	c_3	A-D	P-value
$n=5$	$r=0$	0.65	0.667	0.082	0.60	0.353	0.464	0.66	0.466	0.252
	$r=1$	0.65	0.691	0.071	0.59	0.299	0.586	0.62	0.854	0.028
	$r=0$	0.64	0.229	0.811	0.60	0.600	0.711	0.63	0.339	0.502
	$r=1$	0.65	0.657	0.086	0.61	0.371	0.424	0.68	0.816	0.035
$n=10$	$r=2$	0.65	0.347	0.480	0.62	0.261	0.707	0.65	0.656	0.087
	$r=3$	0.63	0.248	0.751	0.58	0.191	0.898	0.62	0.413	0.338
	$r=0$	0.65	0.338	0.504	0.61	0.570	0.140	0.66	0.292	0.607
	$r=1$	0.67	0.114	0.992	0.61	0.215	0.847	0.61	0.192	0.897
$n=15$	$r=2$	0.67	0.483	0.230	0.62	0.488	0.224	0.63	0.289	0.615
	$r=3$	0.65	0.215	0.850	0.61	0.211	0.859	0.66	0.170	0.933
	$r=0$	0.70	0.340	0.502	0.64	0.266	0.690	0.67	0.757	0.049
$n=20$	$r=1$	0.60	0.384	0.396	0.57	0.338	0.503	0.69	0.236	0.790
	$r=2$	0.67	0.239	0.780	0.64	0.416	0.332	0.65	0.376	0.413
	$r=3$	0.64	0.320	0.533	0.60	0.311	0.553	0.61	0.356	0.458

Table 40. Goodness of fit tests for normality of sampling distributions for some power of $\hat{\sigma}^k_{ne1}$ in the cases of log Weibull distribution via Anderson-Darling test, Ryan-Joiner, Kolmogorov-Smirnov tests.

		c_1	A-D	P-value	RJ	P-value	KS	*p*-value
$n=5$	$r=0$	0.38	3.411	<0.005	1.000	0.088	0.006	<0.010
	$r=1$	0.24	5.495	<0.005	1.000	0.088	0.008	<0.010
	$r=0$	0.41	1.112	0.007	1.000	>0.100	0.004	0.084
	$r=1$	0.42	0.976	0.014	1.000	>0.100	0.004	0.088
$n=10$	$r=2$	0.29	4.605	<0.005	1.000	0.087	0.007	<0.010
	$r=3$	0.18	0.845	0.030	1.000	>0.100	0.004	0.124
	$r=0$	0.46	0.915	0.020	1.000	>0.100	0.004	0.124
	$r=1$	0.49	1.174	<0.005	1.000	>0.100	0.004	0.070
$n=15$	$r=2$	0.44	0.717	0.061	1.000	>0.100	0.004	0.119
	$r=3$	0.36	1.758	<0.005	1.000	0.095	0.005	<0.010
	$r=0$	0.46	1.084	0.008	1.000	>0.100	0.004	0.027
$n=20$	$r=1$	0.46	0.780	0.043	1.000	>0.100	0.004	0.089
	$r=2$	0.51	0.174	0.927	1.000	>0.100	0.002	>0.150
	$r=3$	0.47	0.437	0.296	1.000	>0.100	0.003	>0.150

Table 41. Goodness of fit tests for normality of sampling distributions for some power of $\hat{\sigma}_{ne2}^{k}$ in the cases of log Weibull distribution via Anderson-Darling test, Ryan-Joiner, Kolmogorov-Smirnov tests.

		c_2	A-D	P-value	RJ	P-value	KS	*p*-value
$n=5$	$r=0$	0.43	1.629	<0.005	1.000	0.099	0.006	<0.010
	$r=1$	0.28	6.614	<0.005	1.000	0.087	0.008	<0.010
	$r=0$	0.51	1.559	<0.005	1.000	>0.100	0.006	<0.010
	$r=1$	0.48	1.624	<0.005	1.000	0.099	0.005	<0.010
$n=10$	$r=2$	0.32	3.073	<0.005	1.000	0.092	0.005	<0.010
	$r=3$	0.22	2.835	<0.005	1.000	>0.100	0.005	<0.010
	$r=0$	0.51	0.177	0.922	1.000	>0.100	0.002	>0.150
	$r=1$	0.52	0.304	0.572	1.000	>0.100	0.002	>0.150
$n=15$	$r=2$	0.48	0.401	0.361	1.000	>0.100	0.003	>0.150
	$r=3$	0.35	1.644	<0.005	1.000	0.096	0.005	<0.010
	$r=0$	0.54	0.611	0.112	1.000	>0.100	0.003	>0.150
$n=20$	$r=1$	0.52	0.150	0.963	1.000	>0.100	0.002	>0.150
	$r=2$	0.52	0.548	0.158	1.000	>0.100	0.003	>0.150
	$r=3$	0.48	0.260	0.710	1.000	>0.100	0.002	>0.150

Table 42. The results of estimation for Proschan's data with 95% confidence interval when the population is assumed to be exponentially distributed.

$n=15$	$\hat{\sigma}$	$S_{\hat{\sigma}}$	$\hat{\sigma}_{lower}$	$\hat{\sigma}_{upper}$
$r=0$				
BLUE	121.2667	31.3109		
NE1	139.1612	49.4124	64.9853	264.8628
NE2	120.8899	31.6629	69.1128	194.6605
NE3	127.1332	46.1614	56.4906	240.9225
$r=1$				
BLUE	121.5047	32.4762		
NE1	140.6829	55.1693	59.2043	282.2100
NE2	120.1800	32.1932	67.8757	195.6959
NE3	128.1628	47.6098	55.9451	246.6110
$r=2$				
BLUE	121.1355	33.2844		
NE1	137.8787	61.5902	49.6463	299.1622
NE2	122.8265	34.0143	67.7928	202.8450
NE3	126.9227	51.6077	50.1027	256.9287
$r=3$				
BLUE	86.9075	25.1169		
NE1	90.6245	32.4523	41.0681	171.0186
NE2	86.6153	25.4060	46.0826	147.2096
NE3	81.4125	29.5378	36.3305	154.5069

Table 43. The results of estimation for Proschan's data with 95% confidence interval when the population is assumed to be Rayleigh distributed.

$n=15$	$\hat{\sigma}$	$S_{\hat{\sigma}}$	$\hat{\sigma}_{lower}$	$\hat{\sigma}_{upper}$
$r=0$				
BLUE	121.2745	15.7327		
NE1	120.2222	34.8531	59.1531	194.9853
NE2	119.2181	20.0496	82.5125	160.9840
NE3	119.5211	32.5945	61.9399	188.7493
$r=1$				
BLUE	112.9339	15.1654		
NE1	111.3429	34.5159	51.0320	185.3690
NE2	112.0327	17.8383	79.2495	149.0777
NE3	112.2521	31.9094	56.8208	181.3397
$r=2$				
BLUE	105.2916	14.6801		
NE1	102.9302	32.6351	46.0788	173.0521
NE2	106.1461	18.5927	72.1522	144.9040
NE3	104.1230	32.3260	48.3357	174.2815
$r=3$				
BLUE	67.3348	9.7809		
NE1	65.6438	19.4195	31.7059	107.3704
NE2	66.9899	10.9526	46.8981	89.7688
NE3	67.0067	18.0786	34.9662	105.4664

Table 44. The results of estimation for Proschan's data with 95% confidence interval when the population is assumed to be half logistic distributed.

$n = 15$	$\hat{\sigma}$	$S_{\hat{\sigma}}$	$\hat{\sigma}_{lower}$	$\hat{\sigma}_{upper}$
$r = 0$				
BLUE	90.5335	19.4358		
NE1	104.7203	34.1110	51.1291	186.9134
NE2	89.4004	19.0801	56.6256	131.7125
NE3	89.3517	25.3325	47.8651	147.9298
$r = 1$				
BLUE	89.6988	19.6117		
NE1	101.8838	35.7547	47.0037	190.0588
NE2	88.1975	20.3100	54.0329	134.2190
NE3	87.7139	24.5469	47.3191	144.1914
$r = 2$				
BLUE	87.0062	20.0207		
NE1	97.0438	36.1647	40.3838	183.1611
NE2	86.0486	19.8622	52.1695	130.3118
NE3	85.4292	27.5071	40.9123	149.3138
$r = 3$				
BLUE	62.3036	15.1096		
NE1	63.1212	20.2821	30.3739	110.3692
NE2	59.1785	13.8356	35.6720	90.1356
NE3	59.4911	18.4420	29.4179	102.0669

5. Concluding remarks and discussions

The new scale estimators presented in this study have advantages of the estimators proposed by Ene and Karahasan (2016); first, they are unbiased and have nearly equivalent variance performance with the BLU estimators. Second, unlike BLU estimators, expected values and variance-covariance matrix of order statistics are not needed in computing them. Further, hypotheses tests and confidence interval based on these estimators can easily be constructed because of likely normality of their sampling distributions after some power transformation.

Moreover, the new estimators with weights that involve evaluation of relevant distribution function are put in an equivalent form by using order statistics from standard uniform distribution so that the computation can be easily carried out for the distribution functions that cannot be expressed in a closed form. Therefore, these new unbiased scale parameter estimators can be regarded as an alternative to BLU estimators in the case of no censoring or doubly Type II censoring for the relevant family of scale distributions.

Finally, as extensions of this work, first the estimators can be adapted to Type II right or Type II left censored samples. In addition, NE type new estimators can easily be developed for other scale family of distributions not considered in this work.

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