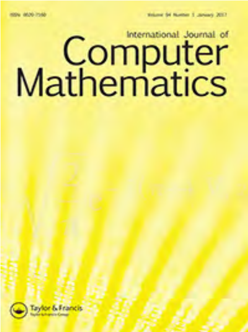

International Journal of Computer Mathematics

ISSN: 0020-7160 (Print) 1029-0265 (Online) Journal homepage: https://www.tandfonline.com/loi/gcom20

Numerical solution the fractional Bagley–Torvik equation arising in fluid mechanics

Mustafa Gülsu, Yalçın Öztürk & Ayşe Anapali

To cite this article: Mustafa Gülsu, Yalçın Öztürk & Ayşe Anapali (2017) Numerical solution the fractional Bagley–Torvik equation arising in fluid mechanics, International Journal of Computer Mathematics, 94:1, 173-184, DOI: 10.1080/00207160.2015.1099633

To link to this article: https://doi.org/10.1080/00207160.2015.1099633

INTERNATIONAL JOURNAL OF COMPUTER MATHEMATICS, 2017
VOL. 94, NO. 1, 173–184
http://dx.doi.org/10.1080/00207160.2015.1099633

Numerical solution the fractional Bagley–Torvik equation arising in fluid mechanics

Mustafa Gülsu[a], Yalçın Öztürk[b] and Ayşe Anapali[a]

[a]Department of Mathematics, Faculty of Science, Muğla Sıtkı Koçman University, Muğla, Turkey; [b]Ula Ali Koçman Vocational School, Muğla Sıtkı Koçman University, Muğla, Turkey

ABSTRACT
In this paper, we present a numerical solution method which is based on Taylor Matrix Method to give approximate solution of the Bagley–Torvik equation. Given method is transformed the Bagley–Torvik equation into a system of algebraic equations. This algebraic equations are solved through by assistance of Maple 13. Then, we have coefficients of the generalized Taylor series. So, we obtain the approximate solution with terms of the generalized Taylor series. Further some numerical examples are given to illustrate and establish the accuracy and reliability of the proposed algorithm.

ARTICLE HISTORY
Received 21 November 2014
Revised 17 July 2015
Accepted 12 September 2015

KEYWORDS
Bagley–Torvik equation; fluid mechanics problem; Taylor matrix method; Taylor series; approximate solution; convergence analysis

1. Introduction

Fractional calculus has become the focus of interest for many researchers in different disciplines of applied science and engineering because of the fact that a realistic modelling of a physical phenomenon, for example viscoelasticity, heat conduction, electrode–electrolyte polarization, electromagnetic waves, diffusion wave, control theory, etc., can be successfully achieved by using fractional calculus[3–5,12,,17,21].

In this article, we consider the Bagley–Torvik equation which is defined by [5,6,14,16,20]

$$(AD^2 + A_3D^{3/2} + A_0D^0)y(x) = f(x) \tag{1}$$

with the initial conditions

$$y(0) = a, y'(0) = b, \tag{2}$$

where $A = m, A_3 = 2A\sqrt{\mu\rho}, A_0 = k$ and where μ is the viscosity, ρ is the fluid density. This equation arises in the modelling of the motion of a rigid plate immersed in a Newtonian fluid. The motion of a rigid plate of mass m and area A connected by a mass less spring of stiffness k, immersed in a Newtonian fluid.

A rigid plate of mass m immersed into an infinite Newtonian fluid as shown in Figure 1. The plate is held at a fixed point by means of a spring of stiffness k. It is assumed that the motions of spring do not influence the motion of the fluid and that the area A of the plate is very large, such that the stress–velocity relationship is valid on both sides of the plate.

CONTACT Yalçın Öztürk yozturk@mu.edu.tr

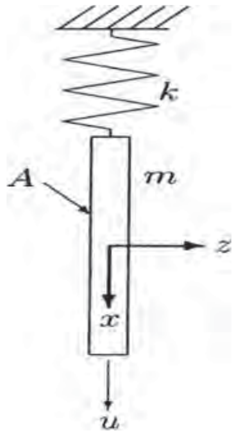

Figure 1. Rigid plate of mass m immersed into a Newtonian fluid.

The existence and uniqueness of the solution to this initial value problem have been discussed in [4,14]. An analytical solution is possible and can be given in the form [16]

$$y(x) = \int_0^x G(x-u)f(u)\mathrm{d}u$$

with

$$G(x) = \frac{1}{A}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{A_0}{A}\right)^k x^{2k+1}E^{(k)}_{1/2,2+3k/2}\left(-\frac{A_3}{A}\sqrt{t}\right),$$

where $E^{(k)}_{\lambda,\mu}$ is the kth derivative of the Mittag–Leffler function with parameters λ and μ given by

$$E^{(k)}_{\lambda,\mu} = \sum_{j=0}^{\infty}\frac{(j+k)!t^j}{j!\Gamma(\lambda j+\lambda k+\mu)}.$$

Note that this analytical solution involves the evaluation of a convolution integral, containing a Green's function expressed as an infinite sum of derivatives of Mittag–Leffler functions, and for general functions f this cannot be evaluated conveniently[16]. For inhomogeneous initial conditions even more complicated expressions arise. An analytical expression for the inhomogeneous case is given in [14]. It involves multivariate generalizations of Mittag–Leffler functions and is also quite cumbersome to handle. We are motivated by the difficulty of obtaining an analytical solution to investigate numerical schemes for the solution of (1) with initial conditions (2) that can be relied upon to perform well. We seek the approximate solution of Equation (1) under the conditions Equation (2) with the generalized Taylor series as $D^{k\alpha}_{*}y(x) \in C(a,b]$,

$$y_N(x) = \sum_{i=0}^{n}\frac{(x-c)^{i\alpha}}{\Gamma(i\alpha+1)}(D^{i\alpha}_{*}y(x))_{x=c}, \tag{3}$$

where $0 < \alpha \leq 1$ [9].

Moreover, some numerical methods have been proposed for approximate solutions of this type equations such as the operational matrix method [8,13,19], Adomian decomposition method [10], homotopy-perturbation method [1], collocation method [2,17] and others [7,11,15,18].

2. Fundamental relations

In this section, we consider the fractional differential equations

$$\sum_{k=0}^{m} P_k(x) D_*^{k\alpha} y(x) = f(x), a \le x \le b, n-1 \le m\alpha < n \tag{4}$$

with initial conditions

$$D_*^{i} y(c) = \lambda_i, i = 0, 1, \ldots, n-1, a \le c \le b, \tag{5}$$

which $P_k(x)$ and $f(x)$ are known functions defined on $a \le x \le b$, λ_i is an appropriate constant.
We first consider the solution $y_N(x)$ of Equation (1) defined by a truncated Taylor series (3). Then, we have the matrix form of the solution $y_N(x)$

$$[y_N(x)] = \mathbf{X}\mathbf{M}_0\mathbf{A}, \tag{6}$$

where

$$\mathbf{X} = [1 \quad (x-c)^{\alpha} \quad (x-c)^{2\alpha} \quad \cdots \quad (x-c)^{N\alpha}]$$

$$\mathbf{M}_0 = \begin{bmatrix} \frac{1}{\Gamma(1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\Gamma(\alpha+1)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\Gamma(2\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\Gamma(N\alpha+1)} \end{bmatrix} \mathbf{A} = \begin{bmatrix} D_*^{0\alpha} y(c) \\ D_*^{1\alpha} y(c) \\ D_*^{2\alpha} y(c) \\ \vdots \\ D_*^{N\alpha} y(c) \end{bmatrix}.$$

Now, we consider the differential part of $P_k(x) D_*^{k\alpha} y(x)$ in Equation (1) and can write it as the truncated Taylor series expansion of degree N at $x = c$ in the form

$$P_k(x) D_*^{k\alpha} y_N(x) = \sum_{n=0}^{N} \frac{1}{\Gamma(n\alpha+1)} [D_*^{i\alpha}(P_k(x) D_*^{k\alpha} y_N(x))]_{x=c} (x-c)^{n\alpha}. \tag{7}$$

By Liebnitz's rule, we evaluate

$$D_*^{n\alpha}[P_k(x) D_*^{k\alpha} y(x)]_{x=c} = \sum_{i=0}^{n} \binom{p}{i} P_k^{(i)}(c) D_*^{n\alpha-i}(D_*^{k\alpha} y_N(x))_{x=c}. \tag{8}$$

Thus expression (7) becomes

$$P_k(x) D_*^{k\alpha} y_N(x) = \sum_{n=0}^{N} \sum_{i=0}^{n} \frac{1}{\Gamma(n\alpha+1)} \frac{N!}{(N-i)!i!} P_k^{(n-i)}(c) [D_*^{(k+i)\alpha}(y_N(x))]_{x=c} (x-c)^{n\alpha} \tag{9}$$

and its matrix form

$$[P_k(x) D_*^{k\alpha} y_N(x)] = \mathbf{X}\mathbf{P}_k\mathbf{A}, \tag{10}$$

where

$$P_k = \begin{bmatrix} 0 & \cdots & 0 & \frac{P_k^{(0)}(c)0!}{\Gamma(1)0!0!} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{P_k^{(1)}(c)1!}{\Gamma(\alpha+1)1!0!} & \frac{P_k^{(0)}(c)1!}{\Gamma(\alpha+1)0!1!} & \cdots & 0 \\ 0 & \cdots & 0 & \frac{P_k^{(2)}(c)2!}{\Gamma(2\alpha+1)2!0!} & \frac{P_k^{(1)}(c)2!}{\Gamma(2\alpha+1)1!!1!} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{P_k^{(N-k)}(c)(N-k)!}{\Gamma((N-k)\alpha+1)(N-k)!0!} & \frac{P_k^{(N-k)}(c)(N-k)!}{\Gamma((N-k)\alpha+1)(N-k-1)!1!} & \cdots & \frac{P_k^{(0)}(c)k!}{\Gamma(k\alpha+1)0!k!} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{P_k^{(N)}(c)N!}{\Gamma(N\alpha+1)N!0!} & \frac{P_k^{(N-1)}(c)N!}{\Gamma(N\alpha+1)(N-1)!1!} & \cdots & \frac{P_k^{(0)}(c)N!}{\Gamma(N\alpha+1)(N-k)!k!} \end{bmatrix}_{(N+1)\times(N+1)}.$$

Moreover, let assume that the function $f(x)$ can be written as a truncated Taylor series

$$f(x) = \sum_{n=0}^{N} \frac{1}{\Gamma(n\alpha+1)} (D_*^{n\alpha} f(x))_{x=c} (x-c)^{n\alpha}. \tag{11}$$

Then, we obtain the matrix form of Equation (11)

$$[f(x)] = \mathbf{X}\mathbf{M}_0\mathbf{F}, \tag{12}$$

where

$$F = [f(c) \quad (D_*^{\alpha} f(x))_{x=c} \quad (D_*^{2\alpha} f(x))_{x=c} \quad \cdots \quad (D_*^{N\alpha} f(x))_{x=c}]^{\mathrm{T}}.$$

Thus, we obtain the fundamental matrix form of Equation (1)

$$\sum_{k=0}^{m} \mathrm{P}_k \mathrm{A} = \mathrm{M}_0 \mathrm{F}. \tag{13}$$

2.1. Matrix representation for the initial conditions

We want to find the matrix representation of $D_*^i y_N(x) = D_*^{(i(1/\alpha))\alpha} y_N(x)$. Then, we write the $P_k(x) = 1$ and $k = (i/\alpha)$ in Equation (7), we obtain the matrix form of the initial conditions as

$$D_*^i y_N(c) = D_*^{(i(1/\alpha))\alpha} y_N(c) = \mathbf{X}(c)\mathbf{M}_k\mathbf{A}, \tag{14}$$

where $\mathbf{M}_k$ is a matrix that $P_k(c) = 1$, others is zero and $k = (i/\alpha)$ in $\mathbf{P}_k$Let us define $\mathbf{U}_i$ as

$$\mathbf{U}_i = \mathbf{X}(c)\mathbf{M}_k = [u_{i0} \quad u_{i1} \quad u_{i2} \quad \cdots \quad u_{iN}] = [\lambda_i], i = 0, 1, \ldots, m-1. \tag{15}$$

3. Method of solution

We can write Equation (11) in the form

$$\mathbf{W}\mathbf{A} = \mathbf{G}, \tag{16}$$

where

$$\mathbf{W} = [w_{ij}] = \sum_{k=0}^{m} \mathbf{P}_k, i, j = 0, 1, \ldots, N \text{ and } \mathbf{G} = \mathbf{M}_0\mathbf{F}.$$

Consequently, to find the unknown Taylor coefficients matrix A, related with the approximate solution of the problem consisting of Equation (1) with conditions (2), by replacing the m row matrices (15) by the last m rows of the matrix (16), we have augmented matrix

$$[\mathbf{W}^*;\mathbf{G}^*]=\left[\begin{array}{cccccc} w_{00} & w_{01} & \cdots & w_{0N} & ; & (D_*^{0\alpha}f(c))/\Gamma(0\alpha+1)\\ w_{10} & w_{11} & \cdots & w_{1N} & ; & (D_*^{1\alpha}f(c))/\Gamma(1\alpha+1)\\ \vdots & \vdots & \ddots & \vdots & ; & \vdots\\ w_{N-m0} & w_{N-m1} & \cdots & w_{N-mN} & ; & (D_*^{(N-m)\alpha}f(c))/\Gamma((N-m)\alpha+1)\\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0\\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1\\ \vdots & \vdots & \ddots & \vdots & ; & \vdots\\ u_{m-10} & u_{m-11} & \cdots & u_{m-1N} & ; & \lambda_{m-1} \end{array}\right]$$

or the corresponding matrix equation

$$\mathbf{W}^*\mathbf{A}=\mathbf{G}^*. \tag{17}$$

So, we obtain to a system of $(N+1)\times(N+1)$ linear algebraic equations with $(N+1)$ unknown coefficients. If $\operatorname{rank}\mathbf{W}^*=\operatorname{rank}[\mathbf{W}^*;\mathbf{G}^*]=N+1$, then we can be write $\mathrm{A}=(\mathrm{W}^*)^{-1}\mathrm{G}^*$. Thus, the matrix A is uniquely determined. Also Equation (1) with conditions (2) has a unique solution. On the other hand, when $|\mathbf{W}^*|=0$, if $\operatorname{rank}W^*=\operatorname{rank}[\mathbf{W}^*;\mathbf{F}^*]<N+1$, then we may find a particular solution. Otherwise if $\operatorname{rank}\mathbf{W}^*\neq\operatorname{rank}[\mathbf{W}^*;\mathbf{F}^*]$, then it is not a solution.

3.1. Accuracy of the solution

To investigate the convergence, we define the error function as:

$$e_N(x)=y(x)-y_N(x), \tag{18}$$

where $y(x)$ and $y_N(x)$ are the exact solution and the computed solution of the Equation (1), respectively. Substituting $y_N(x)$ into Equation (1) leads to

$$(D^2+A_3D^{3/2}+A_0D^0)y_N(x)=f(x)+p_N(x), n\in N, \tag{19}$$

where $p_N(x)$ is the perturbation term that can be obtained by substituting the computed solution $y_N(x)$ into Equation (1), that is,

$$p_N(x)=(D^2+A_3D^{3/2}+A_0D^0)y_N(x)-f(x), n\in N. \tag{20}$$

Now, by subtracting Equation (20) from Equation (1) and using Equation (18), the error function $e_N(x)$ satisfies following relation

$$-p_N(x)=(D^2+A_3D^{3/2}+A_0D^0)e_N(x), n\in N. \tag{21}$$

Theorem 3.1: *f be continuous functions on their domains. Suppose that for some positive M, we have*

$$|D^{(N+1)\alpha}y(x)|\leq M \quad \forall x\in[0,b]. \tag{22}$$

Then,

$$\lim_{N\to\infty}p_N=0.$$

Proof: Suppose that the solution $y(x)$ and computed solution $y_N(x)$ of Equation (1) are approximated by their Taylor expansions about zero. Then we may write

$$e_N(x) = \sum_{k=N+1}^{\infty} \frac{x^{k\alpha}}{\Gamma(k+1)} (D^{k\alpha} y(x))_{x=0}, \tag{23}$$

which can be represented as

$$e_N(x) = \frac{x^{(N+1)\alpha}}{\Gamma(N+1)} (D^{(N+1)\alpha} y(\xi)), \xi \in (0,x) \tag{24}$$

for some $\xi \in (0,x)$ by generalized Taylor theorem.

Replacing $e_N(x)$ by Equation (24) into Equation (21) gives

$$-p_N(x) = (D^2 + A_3 D^{3/2} + A_0 D^0) \frac{x^{(N+1)\alpha}}{\Gamma(N+1)} (D^{(N+1)\alpha} y(\xi)). \tag{25}$$

Therefore, we have

$$|p_N(x)| \leq (D^{(N+1)\alpha} y(\xi)) \frac{\Gamma((N+1)\alpha+1)}{\Gamma(N\alpha+1)}$$
$$\times \left(\frac{b^{(N+1)\alpha-2}}{\Gamma((N+1)\alpha-1)} + A_3 \frac{b^{(N+1)\alpha-3/2}}{\Gamma((N+1)\alpha-(1/2))} + A_0 \frac{b^{(N+1)\alpha}}{\Gamma((N+1)\alpha+1)} \right).$$

We assume $A^* = \max\{1, A_3, A_0\}$ and under assumption, we have

$$|p_N(x)| \leq A^* M \frac{\Gamma((N+1)\alpha+1)}{\Gamma(N\alpha+1)\Gamma(N\alpha+1)\Gamma(N\alpha+1)} b^{(N+1)\alpha}$$

thus, the proof is complete. ■

Theorem 3.2: *Under the assumptions of Theorem 4.1, we have* $\lim_{N\to\infty} e_N = 0$.

Proof: Let assume

$$D^* = D^2 + A_3 D^{3/2} + A_0 D^0.$$

Then, the Equation (21) can be written as

$$D^* e_N(x) = -p_N(x).$$

Under the assumption, $\lim_{N\to\infty} p_N = 0$ and the Equation (1) has unique solution [4,14]. Then, the operator D^* is invertible. Hence $\lim_{N\to\infty} e_N = 0$.

Moreover, we can easily check the accuracy of the method. Since the truncated Taylor series (3) is an approximate solution of Equation (1), when the solution $y(x)$ and its fractional derivatives are substituted in Equation (1), the resulting equation must be satisfied approximately; that is, for $x = x_q \in [a,b],\ q = 0, 1, 2, \ldots$

$$R_N(x_q) = |(D^2 + A_3 D^{3/2} + A_0 D^0) y(x_q) - f(x_q)| \cong 0.$$

■

4. Examples

In order to illustrate the effectiveness of the method proposed in this paper, several numerical examples are carried out in this section.

Example 1: Let us consider the fractional integro-differential equation

$$a_2 D_*^2 y(x) + a_1 D_*^{3/2} y(x) + a_0 y(x) = f(x)$$

with the initial conditions

$$y(0) = 1, y'(0) = 1.$$

Then $f(x) = x + 1$, $a_2 = 1$, $a_1 = 1$, $a_0 = 1$. We seek the approximate solutions y_6 by Taylor series, for $c = 0$ and $\alpha = (1/2)$

$$y_6(x) = \sum_{k=0}^{6} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} (D_*^{k\alpha} y(x))_{x=0}.$$

Fundamental matrix relation of this is

$$(\mathbf{P}_4 + \mathbf{P}_3 + \mathbf{P}_0)\mathbf{A} = \mathbf{G},$$

where

$$\mathbf{P}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2/\sqrt{\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{P}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/\sqrt{\pi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4/3\sqrt{\pi} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{P}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/\sqrt{\pi} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4/3\sqrt{\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8/15\sqrt{\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/6 \end{bmatrix}$$

then, we obtained

$$\mathbf{W} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2/\sqrt{\pi} & 0 & 0 & 2/\sqrt{\pi} & 2/\sqrt{\pi} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4/3\sqrt{\pi} & 0 & 0 & 4/3\sqrt{\pi} \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8/15\sqrt{\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/6 \end{bmatrix}$$

Also, we have the matrix representation of conditions,

$$y(0) = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]\mathbf{A} = [1]$$

$$y(1) = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]\mathbf{A} = [1]$$

then, augmented matrix becomes

$$[\mathbf{W}^*;\mathbf{F}^*]=\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & ;1 \\ 0 & 2/\sqrt{\pi} & 0 & 0 & 2/\sqrt{\pi} & 2/\sqrt{\pi} & 0 & ;0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & ;1 \\ 0 & 0 & 0 & 4/3\sqrt{\pi} & 0 & 0 & 4/3\sqrt{\pi} & ;0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & ;0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & ;1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & ;1 \end{bmatrix}$$

and so we solve the this equation, we obtained the coefficients of the Taylor series

$$\mathbf{A}=[1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0].$$

Hence, for $N=6$, the approximate solution of Example 1 is given $y_6=1+x$ which is the exact solution of this equation.

Example 2: [11] Let us consider the fractional integro-differential equation

$$D_*^2 y(x)+D_*^{3/2}y(x)+y(x)=f(x)$$

with the initial conditions

$$y(0)=0, y(1)=1.$$

Then $f(x)=x^2+2+4\sqrt{x/\pi}$. We seek the approximate solutions y_6 by Taylor series, for $\alpha=(1/2)$

$$y_6(x)=\sum_{k=0}^{6}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}(D_*^{k\alpha}y(x))_{x=0}.$$

Fundamental matrix relation of this is

$$(\mathbf{P}_4+\mathbf{P}_3+\mathbf{P}_0)\mathbf{A}=\mathbf{G},$$

where

$$\mathbf{P}_4=\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2/\sqrt{\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{P}_3=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/\sqrt{\pi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4/3\sqrt{\pi} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{P}_0=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/\sqrt{\pi} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4/3\sqrt{\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8/15\sqrt{\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/6 \end{bmatrix}$$

then, we obtained

$$\mathbf{W} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2/\sqrt{\pi} & 0 & 0 & 2/\sqrt{\pi} & 2/\sqrt{\pi} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4/3\sqrt{\pi} & 0 & 0 & 4/3\sqrt{\pi} \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8/15\sqrt{\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/6 \end{bmatrix}.$$

Also, we have the matrix representation of conditions,

$$y(0) = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]\mathbf{A} = [0]$$
$$y(1) = [1 \quad 2/\sqrt{\pi} \quad 1 \quad 4/3\sqrt{\pi} \quad 1/2 \quad 8/15\sqrt{\pi} \quad 1/6]\mathbf{A} = [1]$$

then, augmented matrix becomes

$$[\mathbf{W}^*; \mathbf{F}^*] = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & ; & 2 \\ 0 & 2/\sqrt{\pi} & 0 & 0 & 2/\sqrt{\pi} & 2/\sqrt{\pi} & 0 & ; & 4/\sqrt{\pi} \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & ; & 0 \\ 0 & 0 & 0 & 4/3\sqrt{\pi} & 0 & 0 & 4/3\sqrt{\pi} & ; & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & ; & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 1 & 2/\sqrt{\pi} & 1 & 4/3\sqrt{\pi} & 1/2 & 8/15\sqrt{\pi} & 1/6 & ; & 1 \end{bmatrix}$$

and so we solve the this equation, we obtained the coefficients of the Taylor series

$$\mathbf{A} = [0 \quad 0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0].$$

Hence, for $N = 6$, the approximate solution of Example 1 is given $y_6 = x^2$ which is the exact solution of this equation.

Example 3: [18] Let us consider the Bagley–Torvik equation

$$D_*^\alpha y(x) + y(x) = 0, 0 < \alpha \leq 2,$$
$$y(0) = 1, y'(0) = 0.$$

If $0 < \alpha \leq 1$, the first initial condition is needed, while all the initial conditions are necessary when $1 < \alpha < 2$. The exact solution of this problem is $y(x) = E_\alpha(-x^\alpha)$. Here $E_\alpha(z)$ denotes the Mittag–Leffler function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

This problem is solved by Taylor matrix method. We obtain the error estimation function for $N = 27$ and $\alpha = 0.5, 1.5$ as, respectively:

$$
\begin{aligned}
R_{27}(x) = {} & 0.2E-14 + 0.2E-14x + 0.2E-14x^2 + 0.4E-14x^{3/2} + 0.1E-14x^3 + 0.6E-14x^{7/2} \\
& + 0.1E-16x^{11/2} + 0.2E-17x^{13/2} - 0.1E-18x^{15/2} + 0.1E-18x^8 \\
& + 0.5E-19x^{17/2} + 0.1E-20x^{19/2} \\
& + 0.1E-20x^{10} + 0.2E-21x^{21/2}0.1E-22x^{23/2} - 0.433044E-10x^{27/2}, \\
R_{27}(x) = {} & 0.2E-14 + 0.4E-14x^{3/2} - 0.1E-18x^{15/2} + 0.1E-19x^9 + 0.21E-21x^{21/2} \\
& + 0.1E-22x^{12} - 0.433044E-10x^{27/2}.
\end{aligned}
$$

We define the maximum errors for $y_N(x)$ as,

$E_N = ||y(x) - y_N(x)||_\infty = \max\{|y(x) - y_N(x)|, x \in [0,1]\}$. In Tables 1 and 2, we give some numerical results such as comparison of maximum absolute errors, maximum error estimation values for $\alpha = 0.5,\ 1.5$, respectively. Moreover, we compare the absolute errors with Operational Matrix Method [18] and Present Method ($N = 27$) in Table 3.

Example 4: Consider the problem [2,15]

$$D_*^2 y(x) + D_*^{3/2} y(x) + y(x) = 8, x \in [0,1]$$

subject to the initial conditions

$$y(0) = 0, y'(0) = 0.$$

Numerical results with comparison to Ref. [15] are given in Table 4.

Table 1. Compare of some numerical values for $\alpha = 0.5$.

N	25	27	30
E_N	10^{-12}	10^{-13}	10^{-16}
$\|\|R_N\|\|_\infty$	10^{-12}	10^{-13}	10^{-15}

Table 2. Compare of some numerical values for $\alpha = 1.5$.

N	25	27	30
E_N	10^{-19}	10^{-24}	10^{-29}
$\|\|R_N\|\|_\infty$	10^{-19}	10^{-24}	10^{-29}

Table 3. Compare of some methods of Example 3.

α	x	Oper. matrix method [2]	Present method ($N = 27$)
0.2	0.1	2.9×10^{-1}	5.6×10^{-7}
	0.3	4.5×10^{-1}	2.2×10^{-5}
	0.5	7.4×10^{-1}	3.7×10^{-3}
	0.7	3.7×10^{-1}	2.7×10^{-2}
	0.9	2.0×10^{-1}	9.5×10^{-2}
0.6	0.1	6.7×10^{-3}	1.0×10^{-14}
	0.3	2.0×10^{-5}	4.0×10^{-14}
	0.5	5.2×10^{-3}	3.0×10^{-14}
	0.7	4.4×10^{-3}	1.0×10^{-14}
	0.9	4.6×10^{-3}	8.0×10^{-14}

Table 4. Comparison of numerical results.

x	Exact solution	Adomian method	Present met. ($N_e = 16$)
0.0	0.000000	0.000000	0.000000
0.2	0.125221	0.140640	0.125254
0.4	0.455435	0.533284	0.455468
0.6	0.950392	1.148840	0.950398
0.8	1.579557	1.963033	1.579689
1.0	2.315526	2.952567	2.315589

5. Conclusion

In this study, we present a Taylor matrix method for the numerical solutions of Bagley–Torvik equation. This method transform Bagley–Torvik equation into a system of linear algebraic equation. The approximate solutions can be obtained by solving the resulting system, which can be effectively computed using symbolic computing codes on Maple 13. This method has been given to find the analytical solutions if the system has exact solutions that are polynomial functions. If the exact solutions of problem are not polynomial functions, then a good approximation can be gained by using the proposed method. Application of the matrix method allows the creation of more effective and faster algorithms than the ordinary ones. This method some considerable advantage of the method is that the Generalized Taylor polynomial coefficients of the solution are found very easily, shorter computation times are so low such as 1.4 sn for Example 3 (CPU Core2 Duo 2.13 Ghz, RAM 2 Gb) and lower operation count results in reduction of cumulative truncation errors and improvement of overall accuracy. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Acknowledgements

The author thanks the editor and reviewers for their suggestions to improve the quality of the paper.

Disclosure statement

No potential conflict of interest was reported by the authors.

References

[1] O. Abdulaziz, I. Hashim, and S. Momani, *Solving systems of fractional differential equations by homotopy-perturbation method*, Phys. Lett. A. 372 (2008), pp. 451–459.
[2] Q.M. Al-Mdallal, M.I. Syam, and M.N. Anwar, *A collocation-shooting method for solving fractional boundary value problems*, Commun. Nonlinear Sci. Numer. Simul. 15(12) (2010), pp. 3814–3822.
[3] R.L. Bagley and P.J. Torvik, *Fractional calculus in the transient analysis of viscoelastically damped structures*, AIAA J. 23(6) (1985), pp. 918–925.
[4] J. Deng and L. Ma, *Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations*, Appl. Math. Lett. 23 (2010), pp. 676–680.
[5] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin Heidelberg, 2010.
[6] K. Diethelm and N.J. Ford, *Numerical solution of the Bagley–Torvik equation*, BIT 42 (2002), pp. 490–507.
[7] K. Diethelm and N.J. Ford, *Numerical solution of the Bagley–Torvik Equation*, Swets Zeitlinger. 42(3) (2002), pp. 490–507.
[8] E.H. Doha, A.H. Bhrawy, and S.S. Ezz-Eldien, *A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order*, Comput. Math. Appl. 62 (2011), pp. 2364–2373.
[9] M. Gülsu and M. Sezer, *A Taylor polynomial approach for solving differential difference equations*, J. Comput. Appl. Math. 186 (2006), pp. 349–364.
[10] Y. Hu, Y. Luo, and Z. Lu, *Analytical solution of the linear fractional differential equation by Adomian decomposition method*, J. Comput. Appl. Math. 215 (2008), pp. 220–229.
[11] Y. Keskin, O. Karaoğlu, S. Servi, and G. Oturaç, *The approximate solution of high order linear fractional differential equations with variable coefficients in terms of generalized Taylor polynomials*, Math. Comput. Appl. 16(3) (2011), pp. 617–629.

[12] R. Lewandowski and B. Chorazyczewski, *Identification of the parameters of the Kelvin–Voigt and the Maxwell fractional models, used to modeling of viscoelastic dampers*, Comput. Struct. 88 (2010), pp. 1–17.

[13] Y. Luchko and R. Gorenflo, *An operational method for solving fractional differential equations with the Caputo derivatives*, Acta Math. Vietnam. 24 (1999), pp. 207–233.

[14] K. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons Inc., New York, 1993.

[15] S. Momani and Z. Odibat, *Numerical comparison of methods for solving linear differential equations of fractional order*, Chaos Solitons Fractals. 31 (2007), pp. 1248–1255.

[16] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[17] E.A. Rawashdeh, *Numerical solution of fractional integro-differential equations by collocation method*, Appl. Math. Comput. 176 (2006), pp. 1–6.

[18] A. Saadatmandi and M. Dehghan, *A new operational matrix for solving fractional order differential equations*, Comput. Math. Appl. 59 (2010), pp. 1326–1336.

[19] S. Saha Ray, *On Haar wavelet operational matrix of general order and its application for the numerical solution of fractional Bagley Torvik equation*, Appl. Math. Comput. 218 (2012), pp. 5239–5248.

[20] P.J. Torvik and R.L. Bagley, *On the appearance of the fractional derivative in the behavior of real materials*, J. Appl. Mech. 51 (1984), pp. 294–298.

[21] B.M. Vinagre, I. Podlubny, A. Hernandez, and V. Feliu, *Some approximations of fractional operators used in control theory and applications*, Fract. Calc. Appl. Anal. 3(3) (2000), pp. 945–950.