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Maximal, irreducible and prime soft ideals of BCK/BCI-algebra

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Abstract

In this paper, the notions of soft irreducible, prime and maximal soft ideals,irreducible, prime and maximal soft idealistic over on BCK/BCI-algebras are introduced, and several examples are given to illustrate. Relations between irreducible, prime and maximal idealistic soft BCK/BCI-algebras are investigated.

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1. Introduction

Dealing with uncertainties is a main problem in many areas such as economics, engineering, environmental science, medical science and social sciences. These kinds of problems cannot be dealt with classical methods. Because, these classical methods have their inherent difficulties. To overcome these kinds of difficulties, Molodtsov [10] proposed a completely new approach, which is called soft set theory, for modeling uncertainty. Molodtsov [10], Jun[6] and Park[7] pointed out several directions for the applications of soft sets. Maji at al[9] studied several operations on the theory of soft sets. Besides, Aktaş and Çağman [1] defined soft groups and obtained the main properties of these groups. Jun[6] applied the notion of soft sets by Molodtsov to the theory of BCK/BCI-algebras. The notion of soft BCK/BCI-algebras and subalgebras introduced, and their basic properties have derived. Then, Jun and Park[7] presented the soft ideals and idealistic soft BCK/BCI-algebras, and their basic properties have given. In this paper we apply the notion of soft sets by Molodtsov to the theory of BCK/BCI-algebras. We introduce the notions of irreducible soft ideals, maximal soft ideals and prime soft ideals of

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BCK/BCI-algebra and irreducible idealistic, prime idealistic and maximal idealistic soft BCK/BCI-algebras and we derive their basic properties.

2. Preliminaries

2.1. Basic results on BCK/BCI-algebras. A BCK- algebra is an important class of logical algebras introduced by K. Iséki[5] and was extensively investigated by several researchers. In this section we give some basic definitions and notions to be used in our work for an easy reference by readers.

We start with a well known definition.

2.1. Definition. Let X be a set with a binary operation $*$ and a constant 0. Then $(X;*,0)$ is called *a BCI-algebra* if it satisfies the following conditions:

BCI-1: $(\forall x, y, z \in X)$ $(((x*y)*(x*z))*(z*y) = 0$,
BCI-2: $(\forall x, y, z \in X)$ $((x*(x*y))*y = 0$,
BCI-3: $(\forall x \in X)$ $(x*x = 0)$,
BCI-4: $(\forall x, y, z \in X)$ $(x*y = 0, y*x = 0 \Rightarrow x = y)$.

2.2. Definition. Let X be a BCI-algebra. If X satisfies the following identity:

BCK-5: $(\forall x \in X)$ $(0*x = 0)$,

then X is called *a BCK-algebra*.

Any BCK-algebra X satisfies the following axioms:

(a1): $(\forall x \in X)$ $(x*0 = x)$
(a2): $(\forall x, y, z \in X)$ $(x \leq y \Rightarrow x*y \leq y*z, z*y \leq z*x$
(a3): $(\forall x, y, z \in X)$ $((x*y)*z = (x*z)*y)$,
(a4): $(\forall x, y, z \in X)$ $((x*z)*(y*z) \leq x*y)$,

where $x \leq y$ if and only if $x*y = 0$. A BCK-algebra X is said to be commutative if x $\wedge$ y = y $\wedge$ x for all $\forall x, y \in X$, where $x \wedge y = y*(y*x)$ is a lower bound of x and y. For any element x of a BCI-algebra X, we define the order of x, denoted by $o(x)$, as

$$o(x) = min\{n \in \mathbb{N} \mid 0*x^n = 0\}$$

2.3. Definition. [8] A nonempty subset S of a BCK/BCI-algebra X is called *a subalgebra* of X if $x*y \in S$ for all $x, y \in S$.

2.4. Definition. [8] A subset H of a BCK/BCI-algebra X is called *an ideal* of X if it satisfies the following axioms:

(I1): $0 \in H$,
(I2): $(\forall x \in X)$ $(\forall y \in H)$ $(x*y \in H \Rightarrow x \in H)$.

Any ideal H of a BCK/BCI-algebra X satisfies the following implication:
$(\forall x \in X)$ $(\forall y \in H)(x \leq y \Rightarrow x \in H)$.
In this study, we present the definition of irreducible soft ideal, prime soft ideal and maximal soft ideal, irreducible, maximal and prime idealistic soft BCK-algebras. On this account, we first recall the definitions of irreducible, prime and maximal ideals on BCK/BCI-algebras.

2.5. Definition. [8] A proper ideal I of a BCK-algebra X is called to be *irreducible* if $I = A \cap B$ implies $I = A$ or $I = B$ for any A, B ideal of X.

2.6. Definition. [8] A BCK-algebra $(X;*,0)$ is said to be a lower BCK-semilattice if X is a lower semilattice with respect to BCK order $\leq$ and denote $x \wedge y = inf\{x, y\}$

2.7. Definition. [8] Let X be a lower BCK-semilattice. A proper ideal I of X is called to be *a prime ideal* if $x \wedge y \in I$ implies $x \in I$ or $y \in I$ for any x, $y \in X$.

2.8. Definition. [8] Given a BCK-algebra $(X;*,0)$, an ideal I of X is called to be *a maximal* ideal if I is a proper ideal of X and not a proper subset of any proper ideal of X.

It is well known that in a commutative BCK-algebra every maximal ideal is a prime ideal, and irreducible and prime ideals coincide. In the next chapter, up to the present, we introduce and investigate soft ideal.

2.2. Basic results on soft sets. Molodtsov defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U and $A \subset E$.

2.9. Definition. [10] A pair (F, A) is called *a soft set over* U, where F is a mapping given by $F : A \longrightarrow P(U)$.

In other words , a soft set over U is a parameterized family of subset of the universe U. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A). Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [3]

2.10. Definition. [10] Let (F, A) and (G, B) be two soft sets over a common universe U. *The intersection* of (F, A) and (G, B) is defined to be the soft set (H, C) satisfying the following conditions:

(i): $C = A \cap B$,
(ii): $(\forall\, e \in C)$ $(H(e) = F(e)$ or $G(e)$, (as both are same sets)).

In this case, we write $(F, A) \mathbin{\widetilde{\cap}} (G, B) = (H, C)$.

2.11. Definition. [10] Let (F, A) and (G, B) be two soft sets over a common universe U. *The union* of (F, A) and (G, B) is defined to be the soft set (H, C) satisfying the following conditions:

(i): $C = A \cup B$

(ii): for all e $\in$ C, $H(e) = \begin{cases} \text{F(e)}, & \text{if e} \in \text{A} \setminus \text{B;} \\ \text{G(e)}, & \text{if e} \in \text{B} \setminus \text{A ;} \\ \text{F(e)} \cup \text{G(e)}, & \text{if e} \in A \cap B. \end{cases}$

In this case, we write $(F, A) \mathbin{\widetilde{\cup}} (G, B)$ $=(H, C)$.

2.12. Definition. [10] If (F, A) and (G, B) be two soft sets over a common universe U, then "(F, A) AND (G, B)" is denoted by $(F, A) \mathbin{\widetilde{\wedge}} (G, B)$ is defined by
$(F, A) \mathbin{\widetilde{\wedge}} (G, B) = (H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$
for all $(\alpha, \beta) \in A \times B$.

2.13. Definition. [3] *The bi-intersection* of two soft sets (F, A) and (G, B) over common universe U is defined to be the soft set (H, C), where $C = A \cap B$ and $H : C \to P(U)$ is a mapping given by $H(x) = F(x) \cap G(x)$ for all $x \in C$. This is denoted by $(F, A) \widetilde{\sqcap} (G, B) = (H, C)$

2.14. Definition. [10] If (F, A) and (G, B) be two soft sets over a common universe U, then "(F, A) OR (G, B)" is denoted by $(F, A) \mathbin{\widetilde{\vee}} (G, B)$ is defined by $(F, A) \mathbin{\widetilde{\vee}} (G, B) = (H, A \times B)$,
where $H(\alpha, \beta) = F(\alpha) \cup G(\beta)$ for all $(\alpha, \beta) \in A \times B$.

2.15. Definition. [10] If (F, A) and (G, B) be two soft set over a common universe U, we say that (F, A) is *a soft subset* of (G, B), denoted by $(F, A) \mathbin{\widetilde{\subset}} (F, A)$, if it satisfies:

i): $A \subset B$,

ii): For every $\varepsilon \in A$, $F(\varepsilon)$ and $G(\varepsilon)$ are same identical approximations.

2.16. Definition. [9] A soft set (F, A) over U is said to be a NULL soft set denoted by Φ, if $\forall\ e \in A,\ F(e) = \emptyset$, (null-set).

2.17. Definition. [9] A soft set over (F, A) over is said to be absolute soft set denoted by $\widetilde{A}$, if $\forall\ e \in A,\ F(e) = U$.

2.18. Definition. [6] Let X be a BCK/BCI-algebra and let (F, A) be a soft set over X. Then (F, A) is called *a soft BCK/BCI-algebra* over X if $F(x)$ is a subalgebra of X for all $x \in A$.

2.19. Definition. [7] Let S be a subalgebra of X. A subset I of X is called *an ideal* of X related to S(briefly, S-ideal of X), denoted by $I_S \lhd S_S$, if it satisfies:

i): $0 \in I$,
i): $(\forall x \in S)(\forall y \in I)\ (x * y \in I) \Rightarrow x \in I$.

2.20. Definition. [7] Let (F, A) be a soft BCK/BCI-algebra over X. A soft set (G, I) over X is called *a soft ideal* of (F, A), denoted by $(G, I)\ \widetilde{\lhd}\ (F, A)$, if it satisfies:

i): $I \subset A$,
ii): $(\forall$ x $\in$ I$)\ (G(x) \lhd F(x))$.

2.21. Definition. [7] Let (F, A) be a soft set over X. Then (F, A) is called *an idealistic soft BCK/BCI-algebra over X* if $F(x)$ is an ideal of X for all $x \in A$ respectively.

3. Maximal, Irreducible and Prime Soft Ideals

We first define soft BCK/BCI subalgebra to complete gap between ideals and algebras. In this section X will be BCK/BCI-algebra.

3.1. Definition. Let (F, A) and (G, B) be two soft BCK/BCI-algebras over X. (G, B) is called *soft BCK/BCI-subalgebra* of (F, A) over X if $B \subseteq A$ and $G(x)$ is a subalgebra of $F(x)$ for each $x \in B$.

3.2. Definition. Let (F, A) be a soft BCK/BCI-algebra over X and (G, I) is a non-whole soft ideal of (F, A).

i) (G, I) is called *an maximal soft ideal* of (F, A) if $G(x)$ is an maximal ideal of $F(x)$ for all $x \in I$.
ii) Then (G, I) is called *an irreducible soft ideal* of (F, A) if $G(x)$ is an irreducible ideal of $F(x)$ for all $x \in I$.

Let (G, I) be a maximal soft ideal of a soft BCK/BCI-algebra (F, A) over X. It is clear that (G, I) is an irreducible soft ideal of (F, A) over X since every maximal ideal of a BCK/BCI-algebra is an irreducible ideal [4, Proposition 3].

3.3. Example. Let $X = \{0, a, b, c, d, e, f, g\}$ be a BCI-algebra with the following Cayley table:

*	0	a	b	c	d	e	f	g
0	0	0	0	0	d	d	d	d
a	a	0	0	0	e	d	d	d
b	b	b	0	0	f	f	d	d
c	c	b	a	0	g	f	e	d
d	d	d	d	d	0	0	0	0
e	e	d	d	d	a	0	0	0
f	f	f	d	d	b	b	0	0
g	g	f	e	d	c	b	a	0

Let (F, A) be a soft set over X where $A = \{0, a, b, c\}$ and $F : A \to P(X)$ is set-valued function defined by $F(x) = \{0\} \cup \{y \in X \mid o(x) = o(y)\}$ for all $x \in A$.
Since $F(0) = F(a) = F(b) = F(c) = \{0, a, b, c\} \leq X$, (F, A) is soft BCI-algebra over X. Let (G, I) be a soft set over X where $I = \{a\}$ and $G : I \longrightarrow P(X)$ is set-valued function defined by $G(x) = \{0, x\}$ for all $x \in I$. Hence $G(a) = \{0, a\} \lhd F(a)$ and ideal $\{0, a\}$ is unique proper ideal of $F(a)$. Thus $G(a)$ is a maximal ideal of $F(a)$ and (G, I) is a maximal soft ideal of (F, A).

3.4. Example. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	c	c	a	0

Let (F, A) be a soft set over X, where $A = X$ and $F : A \to P(X)$ is set-valued function defined by $F(x) = \{y \in X \mid y * x = 0\}$ for all $x \in A$. Then $F(0) = \{0\}$, $F(a) = \{0, a\}$, $F(b) = \{0, a, b\}$, $F(c) = \{0, a, b, c\}$, $F(d) = X$. Then (F, A) is a soft BCK-algebra over X.
Let (G, I) be a soft set over X, where $I = \{c\}$ and $G : I \to P(X)$ is set-valued function defined by $G(x) = \{y \in X \mid y * (y * x) \in \{0, a\}\}$. Then $G(c) = \{0, a\} \lhd F(c)$. Hence (G, I) is a soft ideal of (F, A). Moreover,$I_1 = \{0, a\}$ and $I_2 = \{0, a, b\}$ are ideals of $F(c)$. Then

$$G(c) = I_1 \cap I_2 \ \Rightarrow\ G(c) = I_1.$$

Hence $G(c)$ is an irreducible ideal of $F(c)$. Therefore (G, I) is an irreducible soft ideal of (F, A).

3.5. Example. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	4	4	0

Let (F, A) be a soft set over X where $A = \{1, 2\}$ and $F : A \longrightarrow P(X)$ defined by

$$F(x) = \{y \in X \mid y * x \in \{0, 2, 3\}\}$$

Then we can easly show that

$$\begin{aligned} F(1) &= \{y \in X \mid y * 1 \in \{0, 2, 3\}\} = \{0, 1, 2, 3\} \\ F(2) &= \{y \in X \mid y * 2 \in \{0, 2, 3\}\} = \{0, 2, 3\} \end{aligned}$$

Since $F(1)$ and $F(2)$ are subalgebras of X, (F, A) is a soft BCK-algebra over X. We define soft set (G, B) with $G(x) = \{y \in X \mid y * x = 0\}$ where $B = \{1\}$ and $G : B \longrightarrow P(X)$. So

$$G(1) = \{y \in X \mid y * 1 = 0\} = \{0, 1\}$$

and $G(1)$ is an ideal of $F(1)$. The set of all ideals of $F(1)$ is
$\{I_0 = \{0\}, I_1 = \{0, 1\}, I_2 = \{0, 2\}, I_3 = \{0, 3\}, I_4 = \{0, 1, 2\}, I_5 = \{0, 1, 3\},$
$I_6 = \{0, 2, 3\}, I_7 = F(1)\}$. Thus we see $G(1) = I_4 \cap I_5$ but $G(1) \neq I_4$ and $G(1) \neq I_5$. That is $G(1)$ is not irreducible ideal of $F(1)$. Hence (G, B) is not irreducible soft ideal of (F, A).

The authors Jun et al.[7, Theorem 4.6] proved that intersection of soft ideals is a soft ideal. The following theorem shows that intersection of soft irreducible ideals is a soft irreducible ideal.

3.6. Theorem. *Let (F, A) be a soft BCK/BCI-algebra over X. Let (G_1, I_1) and (G_2, I_2) be two irreducible ideals of (F, A). If $I_1 \cap I_2 \neq \emptyset$, then $(G_1, I_1) \mathbin{\widetilde{\cap}} (G_2, I_2)$ is an irreducible ideal of (F, A) .*

Proof. Using Definition 2.10, we can write $(G_1, I_1) \mathbin{\widetilde{\cap}} (G_2, I_2) = (G, I)$ where $I = I_1 \cap I_2$ and $G(x) = G_1(x)$ or $G(x) = G_2(x)$ for all $x \in I$. We know that $G_1(x) \lhd F(x)$ or $G_2(x) \lhd F(x)$ for all $x \in I$. Hence

$$(G_1, I_1) \mathbin{\widetilde{\cap}} (G_2, I_2) = (G, I) \widetilde{\lhd} (F, A).$$

Since (G_1, I_1) *and* (G_2, I_2) are irreducible ideals of (F, A), we have that $G(x) = G_1(x)$ is an irreducible ideal of $F(x)$ or $G(x) = G_2(x)$ is an irreducible ideal of $F(x)$ for all $x \in I$. Hence $(G_1, I_1) \mathbin{\widetilde{\cap}} (G_2, I_2) = (G, I)$ is an irreducible soft ideal of (F, A). This complete the proof. □

3.7. Theorem. Let (F, A) be a soft BCK/BCI-algebra over X. Let (G, B) *and* (H, C) be two irreducible ideals of (F, A). If $B \cap C = \emptyset$ then $(G, B) \mathbin{\widetilde{\cup}} (H, C)$ is an irreducible ideal of (F, A).

Proof. Let $D = B \cup C$. Define T on D by

$$T(x) = \begin{cases} \text{G(x)}, & \text{if x} \in \text{B} \setminus \text{C} \\ \text{H(x)}, & \text{if x} \in \text{C} \setminus \text{B} \end{cases}$$

Then it is easily checked that $(G, B) \mathbin{\widetilde{\cup}} (H, C) = (T, D)$ and $(T, D) \widetilde{\lhd} (F, A)$. Since $B \cap C = \emptyset$, either $x \in B \backslash C$ or $x \in C \backslash B$ for all $x \in D$. Let $x \in D$. If $x \in B \backslash C$, then $T(x) = G(x)$ is an irreducible ideal of $F(x)$ since (G, B) is an irreducible soft ideal of (F, A). If $x \in C \backslash B$, $T(x) = H(x)$ is an irreducible ideal of $F(x)$ since (H, C) is an irreducible soft ideal of (F, A). Hence $T(x)$ is an irreducible ideal of $F(x)$ for all $x \in D$, and so $(G, B) \mathbin{\widetilde{\cup}} (H, C)$ is an irreducible ideal of (F, A). □

4. Irreducible idealistic soft BCK/BCI-algebras

4.1. Definition. Let (F, A) be a non-whole an idealistic soft BCK/BCI-algebra over X. Then (F, A) is called *an irreducible idealistic soft BCK/BCI-algebra over X* if $F(x)$ is an irreducible ideal of X for all $x \in A$.

4.2. Example. Let $X = \{0, a, b, c, d, e, f, g\}$ and consider the following Cayley table:

*	0	a	b	c	d	e	f	g
0	0	0	0	0	d	d	d	d
a	a	0	0	0	e	d	d	d
b	b	b	0	0	f	f	d	d
c	c	b	a	0	g	f	e	d
d	d	d	d	d	0	0	0	0
e	e	d	d	d	a	0	0	0
f	f	f	d	d	b	b	0	0
g	g	f	e	d	c	b	a	0

Then $(X; 0, *)$ is a BCI-algebra (see [2]). Let (F, A) be a soft set over X, where $A = \{0, a, b, c\}$ and $F : A \to P(X)$ is a set-valued function defined by :

$$F(x) = \{0\} \cup \{y \in X \mid o(x) = o(y)\} \text{ for all } x \in A.$$

Then $F(0) = F(a) = F(b) = F(c) = \{0, a, b, c\}$ is an irreducible ideal of X. Hence (F, A) is an irreducible idealistic soft BCI-algebra over X.

4.3. Example. Let $X = \{0, 1, 2, 3\}$ be a BCK-algebras with Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

Let (F, A) be a soft set over X, where $A = \{0, 1, 2\}$ and $F : A \to P(X)$ be a set-valued function defined by $F(x) = \{y \in X \mid y * x = 0\}$ for all $x \in A$. Then $F(0) = \{0\}$, $F(1) = \{0, 1\}$, $F(2) = \{0, 2\}$ which are irreducible ideals of X. Hence (F, A) is an irreducible idealistic soft BCK-algebra over X.

4.4. Example. Let $X = \{0, 1, 2, 3\}$ be a BCK-algebra with the following Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	2
3	3	3	3	0

Let (F, A) be a soft set over X , where $A = X$ and $F : A \longrightarrow P(X)$ is set-valued function defined by

$$F(x) = \{y \in X \mid y * x = 0\}$$

for all $x \in A$. Then $F(0) = \{0\}$, $F(1) = \{0, 1\} \lhd X$, $F(2) = \{0, 1, 2\} \lhd X$, $F(3) = \{0, 1, 3\} \lhd X$.Therefore (F, A) is idealistic soft BCK-algebra over X. Moreover $F(1) = \{0, 1, 2\} \cap \{0, 1, 3\}$ but $F(1) \neq \{0, 1, 2\}$ and $F(1) \neq \{0, 1, 3\}$, and so $F(1)$ is not irreducible ideal of X. Hence (F, A) is not irreducible idealistic soft BCK-algebra over X.

Note that "AND" of two irreducible soft ideals may not be an irreducible soft ideal.To prove this idea, we examine the following example.

4.5. Example. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

(i) Let (H, B) be a soft set over X, where $B = \{1, 2, 4\}$ and $H : B \longrightarrow P(X)$ be a set-valued function defined by

$$H(x) = \{y \in X \mid y * (y * x) = 0\}$$

for all $x \in B$. Then $H(1) = \{0, 2, 4\}$, $H(2) = \{0, 1, 4\}$ and $H(4) = \{0, 1, 2, 3\}$ which are irreducible ideal of X. Therefore (H, B) is an irreducible idealistic soft BCK-algebra over X.

(ii) Let (K, C) be a soft set over X, where $C = \{2\}$ and $K : C \longrightarrow P(X)$ be a set-valued function defined by

$$K(x) = \{y \in X \mid y * x \in \{0, 1\}\}$$

for all $x \in C$. Then $K(2) = \{0,1,2,3\}$ which is irreducible ideal of X. Hence (K,C) is an irreducible idealistic soft BCK-algebra over X.
Use ii) and iii), $(H,B)\widetilde{\wedge}(K,C) = (F,D)$, $D = B \times C = \{(1,2),(2,2),(4,2)\}$ and $F(1,2) = H(1) \cap K(2) = \{0,2\} \lhd X$, $F(2,2) = H(2) \cap K(2) = \{0,1\} \lhd X$, $F(2,4) = H(4) \cap K(2) = \{0,1,2,3\} \lhd X$. Then
$F(1,2) = \{0,2\} = \{0,1,2,3\} \cap \{0,2,4\}$ and $F(1,2) \neq \{0,1,2,3\}$ and $F(1,2) \neq \{0,2,4\}$. Hence $(H,B)\widetilde{\wedge}(K,C) = (F,D)$ is not irreducible idealistic soft BCK-algebra over X.

4.6. Theorem. *Let (F,A) and (G,B) be two irreducible idealistic soft BCK/BCI-algebras over X. If $A \cap B \neq \emptyset$, then the intersection $(F,A) \mathbin{\widetilde{\cap}} (G,B)$ is an irreducible idealistic soft BCK/BCI-algebras over X.*

Proof. Let $x \in C = A \cap B$. By Definition 2.10, for all $x \in C$, we have $H(x) = F(x)$ or $H(x) = G(x)$ or $H(x) = F(x) = G(x)$ if $F(x) = G(x)$. For all $x \in C$, $F(x)$ and $G(x)$ are irreducible, so $H(x)$ is irreducible too. This completes the proof. □

4.7. Theorem. *Let (F,A) and (G,B) be two irreducible idealistic soft BCK/BCI-algebras over X. If A and B are disjoint, then the union $(H,C) = (F,A) \mathbin{\widetilde{\cup}} (G,B)$ is an irreducible idealistic soft BCK/BCI-algebras over X.*

Proof. Let $C = A \cup B$, $A \cap B = \phi$ be null set and $(H,C) = (F,A)\widetilde{\cup}(G,B)$. By hypothesis $F(a)$ is an irreducible ideal of X for all $a \in A$ and $G(b)$ is an irreducible ideal of X for all $b \in B$. Let $c \in C$. Since $A \cap B = \phi$, $c \in A \setminus B$ or $c \in B \setminus A$. By Definition 2.11, if $c \in A \setminus B$, then $H(c) = F(c)$ is irreducible ideal. If $c \in B \setminus A$, then $H(c) = G(c)$ is an irreducible ideal of X. By definition $H(c)$ is an irreducible ideal of X also. This completes the proof. □

5. Prime Idealistic Soft BCK/BCI-algebras

In this section, we introduce the definition of prime soft idealistic over lower BCK-semilattice and several example are given. X will be a lower BCK-semilattice throughout this section. We start some definitions

5.1. Definition. Let X be lower BCK-semilattice and (F,A) be a soft BCK/BCI-algebra over X and (G,I) is a soft ideal of (F,A). (G,I) is a non-whole called *a prime soft ideal* of (F,A) if $G(x)$ is a prime ideal of $F(x)$ for all $x \in I$.

5.2. Definition. Let (F,A) be a soft idealistic BCK/BCI-algebra over X. (F,A) is called *a prime idealistic soft BCK/BCI-algebra over X* if $F(x)$ is a prime ideal of X for all $x \in A$.

5.3. Example. Let X={0,a,b,c,d} be a lower BCK-semilattice with following Cayley table:

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Let (F,A) be a soft over X, where $A = \{b,c,d\}$ and $F : A \to P(X)$ is a set-valued function defined by $F(x) = \{y \in X \mid y \in x^{-1}I\}$ for all $x \in A$ where $I = \{0,a\} \subset X$ and $x^{-1}I = \{y \in X \mid x \wedge y \in I\}$. Then $F(b) = \{0,a,c,d\}$, $F(c) = \{0,a,b,d\}$ and $F(d) = \{0,a,b,c\}$ are prime ideals of X. Therefore (F,A) is a prime soft idealistic BCK/BCI-algebra over X.

5.4. Theorem. *Let (F, A) and (G, B) be two prime idealistic soft BCK/BCI-algebras over X. If $A \cap B \neq \emptyset$, then the intersection $(H, C) = (F, A) \widetilde{\cap} (G, B)$ is a prime idealistic soft BCK/BCI-algebras over X.*

Proof. Let $x \in C = A \cap B$. By Definition 2.10, for all $x \in C$, we have $H(x) = F(x)$ or $H(x) = G(x)$ or $H(x) = F(x) = G(x)$ if $F(x) = G(x)$. Since, for all $x \in C$, $F(x)$ and $G(x)$ are prime, it follows that for all $x \in C$, $H(x)$ is prime. This completes the proof. □

5.5. Theorem. *Let (F, A) and (G, B) be two prime idealistic soft BCK/BCI-algebras over X. If A and B are disjoint, then the union $(H, C) = (F, A) \widetilde{\cup} (G, B)$ is a prime idealistic soft BCK/BCI-algebras over X.*

Proof. Let $C = A \cup B$, $A \cap B = \phi$ be null set and $(F, A)\widetilde{\cup}(G, B)$. By hypothesis $F(a)$ is a prime ideal of X for all $a \in A$ and $G(b)$ is a prime ideal of X for all $b \in B$. Let $c \in C$. Then $c \in A$ or $c \in B$ but not c belongs to $A \cap B$. By Definition 2.11, if $c \in A \setminus B$, then $H(c) = F(c)$ is a prime ideal. If $c \in B \setminus A$, then $H(c) = G(c)$ is a prime ideal of X. Hence $H(x)$ is a prime ideal of X for all $x \in C$. This completes the proof. □

Example 5.6 shows that the bi-intersection of two prime idealistic soft BCK/BCI-algebras (F, A) and (G, B) over a set X need not be a prime idealistic soft BCK/BCI-algebra.

5.6. Example. Let $X = \{0, a, b, c\}$ be a lower BCK-semilattice with following Cayley table:

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Let $A = \{b, c\}$, $B = \{c\}$ and $I = \{0, a\}$, $I' = \{0, b\}$. Define $F : A \to \mathcal{P}(X)$ by, for $x \in A$, $F(x) = \{y \mid x \wedge y \in I\}$; similarly $G : B \to \mathcal{P}(X)$ by for $x \in B$, $G(x) = \{y \mid x * y \in I'\}$. Then $F(b) = F(c) = \{0, a\}$ and $G(c) = \{0, c\}$. By construction of F, $F(b)$, $F(c)$ and $G(c)$ are prime idealistic soft BCK-algebra then $F(c) \cap G(c) = 0$ and $a \wedge b = b * (b * a) = b * b = 0 \in F(c) \cap G(c)$. But neither a nor b belongs to $F(c) \cap G(c)$. It follows that bi-intersection of two prime idealistic soft BCK/BCI-algebras (F, A) and (G, B) over X need not be a prime idealistic soft BCK/BCI-algebra.

6. Maximal Idealistic Soft BCK/BCI-algebra

In this section, we introduce the definition of maximal idealistic soft BCK/BCI-algebra and several example are given. Moreover, we construct some basic properties using the definition of irreducible idealistic soft algebra, irreducible idealistic soft BCK/BCI-algebra and maximal idealistic soft BCK/BCI-algebra over X.

6.1. Definition. Let (F, A) be a soft idealistic BCK/BCI-algebra over X. (F, A) is called *a maximal idealistic soft BCK/BCI-algebra over X* if $F(x)$ is a maximal ideal of X for all $x \in A$.

6.2. Example. Let $X = \{0, a, b, c, d, e, f, g\}$ and consider the following Cayley table:

*	0	a	b	c	d	e	f	g
0	0	0	0	0	d	d	d	d
a	a	0	0	0	e	d	d	d
b	b	b	0	0	f	f	d	d
c	c	b	a	0	g	f	e	d
d	d	d	d	d	0	0	0	0
e	e	d	d	d	a	0	0	0
f	f	f	d	d	b	b	0	0
g	g	f	e	d	c	b	a	0

Then $(X;0,*)$ is a BCI-algebra (see[2]). Let (F,A) be a soft set over X, where $A = \{0,a,b,c\}$ and $F : A \to P(X)$ is a set-valued function defined as follows: $F(x) = \{0\} \cup \{y \in X \mid o(x) = o(y)\}$ for all $x \in$ A.
Then $F(0) = F(a) = F(b) = F(c) = \{0,a,b,c\}$ is an maximal ideal of X. Hence (F,A) is a maximal idealistic soft BCI-algebra over X.

We know that every maximal ideal is an irreducible ideal on BCK-algebra[4], and prime ideals and irreducible ideals are the same in a commutative BCK-algebra which is clear from [8, Theorem 8.10]. We will give the same properties on soft BCK/BCI-algebras.

6.3. Theorem. Every maximal idealistic soft BCK/BCI-algebra over X is an irreducible idealistic soft BCK/BCI-algebra over X.

Proof. Let (F,A) be a maximal soft idealistic BCK/BCI-algebra over X and let I and J be ideals of X such that $F(x) = I \cap J$ for all $x \in A$. Then $F(x) \subseteq I$ and $F(x) \subseteq J$. Then $F(x) = I$ or $F(x) = J$ since $F(x)$ is a maximal ideal of X. Hence $F(x)$ is an irreducible ideal of X. Therefore (F,A) is an irreducible soft idealistic BCK/BCI-algebra over X. □

6.4. Example. Let X={0,a,b,c,d} be a BCK-algebra with following Cayley table:

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Let (F,A) be a soft over X, where $A = \{b,c,d\}$ and $F : A \to P(X)$ is a set-valued function defined by
$F(x) = \{y \in X \mid y \in x^{-1}I\}$ for all $x \in A$ where $I = \{0,a\} \subset X$ and $x^{-1}I = \{y \in X \mid x \wedge y \in I\}$. Then $F(b) = \{0,a,c,d\}$, $F(c) = \{0,a,b,d\}$ and $F(d) = \{0,a,b,c\}$ are maximal ideals of X. Therefore (F,A) is a maximal soft idealistic BCK/BCI-algebra over X.

6.5. Theorem. Let X be a commutative BCK/BCI-algebra. Then (F,A) is a prime idealistic soft BCK/BCI-algebra over X if and only if it is an irreducible idealistic soft BCK/BCI-algebra over X.

Proof. Let (F,A) be a prime idealistic soft BCK/BCI-algebra over X. , $F(x)$ is a prime ideal for each $x \in A$ if and only if $F(x)$ is an irreducible ideal since X is commutative by [8, Theorem 8.10]. □

Every prime idealistic soft BCK-algebra need not be a maximal idealistic soft BCK-algebra over commutative algebra X.

6.6. Example. Let $X = \{0,1,2,3,4\}$ be commutative BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	4	4	4	0

Let (F, A) be a soft set over X where $A = \{1, 2\}$ and $F : A \longrightarrow P(X)$ is defined by, for $x \in A$

$$F(x) = \{y \in X \mid y * x \in \{0, 1\}\}$$

Therefore $F(1) = \{0, 1, 2\}$ and $F(2) = \{0, 1, 2\}$ are prime ideals of X but not maximal ideals. Hence (F, A) is prime soft idealistic but not maximal soft idealistic. In fact $\{0, 1, 2, 3\}$ is an ideal and properly contain $\{0, 1, 2\}$.

7. Direct Products

Let $(X; *_1; 0)$ and $(Y; *_2; 0)$ be two algebras of type (2,0) and $X \times Y$ denote the direct product set of sets X and Y. Define $*$ operation on $X \times Y$ with

$(a, b) * (c, d) = (a *_1 c,\ b *_2 d)$ where $(a, b),\ (c, d) \in X \times Y$ .... (*)

Let 0 denote $(0, 0) \in X \times Y$. Then it is easy to check that $(X \times Y; *; 0)$ becomes a BCK/BCI-algebra. Namely $(X \times Y; *; 0)$ satisfies **BCI-1** : Let $(x, y), (a, b), (u, v) \in X \times Y$. Then
$(((x, y) * (a, b)) * ((x, y) * (u, v)) * ((u, v) * (a, b)) =$
$(((x *_1 a, y *_2 b) * (x *_1 u, y *_2 v)) * (u *_1 a, v *_2 b) =$
$((((x *_1 a) *_1 (x *_1 u), (y *_2 b) *_2 (y *_2 v)) * (u *_1 a, v *_2 b) =$
$((((x *_1 a) *_1 (x *_1 u)) *_1 (u *_1 a), ((y *_2 b) *_2 (y *_2 v)) *_2 (v *_2 b)) = 0$. The other conditions for $(X \times Y; *; 0)$ to be a BCK/BCI-algebra are satisfied similarly.

7.1. Definition. The BCK/BCI-algebra $(X \times Y; *; 0)$ with (*) operation is called *direct product BCK/BCI-algebra* of BCK/BCI-algebras X and Y.

7.2. Lemma. [8]. *Let S be a subalgebra of $(X; *_1; 0)$ and T be a subalgebra of $(Y; *_2; 0)$. Then $S \times T$ is a subalgebra of $X \times Y$*

Note that from now on we use xy for the element $x*y$ obtained by the binary operation $*$.

7.3. Definition. Let $X \times Y$ be a direct product BCK/BCI-algebra of BCK/BCI-algebras X and Y, and let (F, A) be a soft set over X and (G, B) be a soft set over Y. Define

$$F \times G : A \times B \to P(U) \times P(U)$$

$$\text{by } (F \times G)(a, b) = F(a) \times G(b) \text{ where } (a, b) \in A \times B.$$

It is easy to check that $F \times G$ is well-defined. We call $(F \times G, A \times B)$ *direct product soft set* of the soft sets (F, A) and (G, B).

7.4. Theorem. *Let (F, A) be a soft BCK/BCI-algebra over X and (G, B) be a soft BCK/BCI-algebra over Y. Then $(F \times G, A \times B)$ is a soft BCK/BCI-algebra over $X \times Y$.*

Proof. Let $(a, b) \in A \times B$. By definition $(F \times G)(a, b) = F(a) \times G(b)$ is a direct product of subalgebras $F(a)$ and $G(b)$ of X and Y respectively. Hence $F(a) \times G(b)$ is a subalgebra of $X \times Y$ by Lemma 7.2. □

7.5. Lemma. *Let X and Y be BCK/BCI-algebras and H_1 and H_2 be ideals of BCK/BCI-algebras X and Y respectively. Then $H_1 \times H_2$ are ideals of BCK/BCI-algebra $X \times Y$.*

Proof. Let $(x, y) \in X \times Y$ and $(h_1, h_2) \in H_1 \times H_2$. Assume that $(x, y)(h_1, h_2) \in H_1 \times H_2$. Then $(x, y)(h_1, h_2) = (xh_1, yh_2) \in H_1 \times H_2$ implies $xh_1 \in H_1$ and $yh_2 \in H_2$. Then $x \in H_1$ and $y \in H_2$ since H_1 and H_2 are ideals of BCK/BCI-algebras X and Y respectively. Hence $(x, y) \in X \in H_1 \times H_2$. This completes the proof. □

7.6. Theorem. *Let (F, A) be a soft BCK/BCI-algebra over X and (G, B) be a soft BCK/BCI-algebra over Y. If (F_1, A_1) is a soft ideal of (F, A) and (G_1, B_1) is a soft ideal of (G, B), then $(F_1 \times G_1, A_1 \times B_1)$ is a soft ideal of $(F \times G, A \times B)$.*

Proof. Let (F_1, A_1) be a soft ideal of (F, A) and (G_1, B_1) be a soft ideal of (G, B). Then $A_1 \subseteq A$ and $B_1 \subseteq B$ and $F_1(x)$ is ideal of $F(x)$ for all $x \in A_1$ and $G_1(y)$ is ideal of $G(y)$ for all $y \in B_1$. We first fix $(x, y) \in A_1 \times B_1$ and let $(a, b) \in F(x) \times G(y)$ and $(m, n) \in F_1(x) \times G_1(y)$ where $a \in F(x), b \in G(y), m \in F_1(x), n \in G_1$. Assume that $(a, b)(m, n) \in F_1(x) \times G_1(y)$. By definition $am \in F_1(x), bn \in G_1(y)$. Since $F_1(x)$ and $G_1(y)$ are ideals of $F(x)$ and $G(y)$, we have $a \in F_1(x)$ and $b \in G_1(y)$. Hence $(a, b) \in F_1(x) \times G_1(y)$. □

7.7. Theorem. *Let (F, A) be a idealistic soft BCK/BCI-algebra over X and (G, B) be a idealistic soft BCK/BCI-algebra over Y. Then $(F \times G, A \times B)$ is a idealistic soft BCK/BCI-algebra over $X \times Y$.*

Proof. By Definition 2.21, to complete the proof we show that, for any $(a, b) \in A \times B$, $F(a) \times G(b)$ is a ideal of $X \times Y$ since, by definition $(F \times G)(a, b) = F(a) \times G(b)$. $F(a)$ and $G(b)$ is an ideal of X and Y respectively. So for any $x \in X$, $y \in Y$ and $m \in F(a)$, $n \in G(b)$, $xm \in F(a)$ implies $x \in F(a)$. Similarly $yn \in G(b)$ implies $y \in G(b)$. Hence Assume that $(x, y)(m, n) \in (F \times G)(a, b)$. Then $(xm, yn) = (x, y)(m, n) \in (F \times G)(a, b) = (F(a), G(b))$ implies $xm \in F(a)$ and $yn \in G(b)$. Since $F(a)$ and $G(b)$ are ideals in X and Y respectively, we have $x \in F(a)$ and $y \in G(b)$. Hence $(x, y) \in F(a) \times G(b) = (F \times G)(a, b)$. This completes the proof. □

7.8. Theorem. *Let (F, A) be an irreducible idealistic soft BCK/BCI-algebra over X and (G, B) be an irreducible idealistic soft BCK/BCI-algebra over Y. If, for any $(a, b) \in A \times B$, $F(a) \times G(b)$ are irreducible ideals of $X \times Y$, then $F(a)$ and $G(b)$ are irreducible ideals of X and Y. The converse is true if for any ideal I of $F(a) \times G(b)$ has the form $I_1 \times I_2$ for some ideals I_1 of $F(a)$ and I_2 of $G(b)$ for all $(a, b) \in A \times B$.*

Proof. Let I_1, I_2 be two ideals of X and I'_1, I'_2 be two ideals of Y. We first prove that $(I_1 \cap I_2) \times (I'_1 \cap I'_2) = (I_1 \times I'_1) \cap (I_2 \times I'_2)$. For $(a, b) \in (I_1 \cap I_2) \times (I'_1 \cap I'_2)$ if and only if $a \in I_1 \cap I_2$, $b \in I'_1 \cap I'_2$ if and only if $(a, b) \in (I_1 \times I'_1) \cap (I_2 \times I'_2)$.
Assume that $F(a) \times G(b)$ are irreducible ideals of $X \times Y$. Let $F(a) = I_1 \cap I_2$ and $G(b) = I'_1 \cap I'_2$. Then $F(a) \times G(b) = (I_1 \cap I_2) \times (I'_1 \cap I'_2)$. From what we have proved preceding, it follows that $F(a) \times G(b) = (I_1 \times I'_1) \cap (I_2 \times I'_2)$. Then $F(a) \times G(b) = I_1 \times I'_1$ or $F(a) \times G(b) = I_2 \times I'_2$. Hence $(F(a) = I_1, G(b) = I'_1)$ or $(F(a) = I_2, G(b) = I'_2)$. It implies that $F(a) \times G(b) = I_1 \times I'_1$ or $F(a) \times G(b) = I_2 \times I'_2$.
Conversely, assume that $F(a)$ and $G(b)$ are irreducible ideals for all $a \in A$ and $b \in B$. Let $F(a) \times G(b) = I \cap J$ for some ideals I and J. By hypothesis $I = I_1 \times I'_1$ and $J = J_2 \times J'_2$. Then $F(a) \times G(b) = (I_1 \times I'_1) \cap (J_2 \times J'_2) = (I_1 \cap J_2) \times (I'_1 \cap J'_2)$. Then $F(a) = I_1 \cap J_2$ and $G(b) = I'_1 \cap J'_2$. By assumption $F(a) = I_1 = J_2$ and $G(b) = I'_1 = J'_2$. Hence $F(a) \times G(b) = I = J$ □

Example 7.9 shows that the converse of Theorem 7.8 need not be true in general.

7.9. Example. Let $X = \{0, 1, 2, 3\}$ be the algebra and the irreducible ideals $F(1) = \{0, 1\}$ and $F(2) = \{0, 2\}$ considered in Example 4.3. Consider the sets

$I_1 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}$ and $I_2 = \{(0,0),(0,2),(0,3),(1,0),(1,2),(1,3)\}$. It is easy to check that I_1 and I_2 are ideals and $F(1)\times F(2) = \{(0,0),(0,1),(1,0),(1,2)\} = I_1 \cap I_2$. But $F(1) \times F(2) \neq I_1$ and $F(1) \times F(2) \neq I_2$.

8. Conclusion

In this article, we introduced the notion of soft irreducible, prime and maximal soft ideals, irreducible, prime and maximal soft idealistic over BCK/BCI-algebras. Moreover, we presented to prove the important theorems of classical algebra for soft BCK/BCI-algebras in Theorem6.3 and Theorem6.5.

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