Research Article

A Collocation Method for Solving Fractional Riccati Differential Equation

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We have introduced a Taylor collocation method, which is based on collocation method for solving fractional Riccati differential equation with delay term. This method is based on first taking the truncated Taylor expansions of the solution function in the fractional Riccati differential equation and then substituting their matrix forms into the equation. Using collocation points, we have the system of nonlinear algebraic equation. Then, we solve the system of nonlinear algebraic equation using Maple 13, and we have the coefficients of the truncated Taylor sum. In addition, illustrative examples are presented to demonstrate the effectiveness of the proposed method. Comparing the methodology with some known techniques shows that the present approach is relatively easy and highly accurate.

1. Introduction

The concept of fractional or noninteger order derivation and integration can be traced back to the genesis of integer order calculus itself [1, 2]. The recent investigations in science and engineering have demonstrated that the dynamics of many systems may be described more accurately by using differential equations of noninteger order. The fractional differential equations FDEs have shown to be adequate models for various physical phenomena in areas like damping laws, diffusion processes, and so forth. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives [3] and the fluid-dynamic traffic model with fractional derivatives [4], psychology [5] and so forth, [6–9].

In this paper, we present numerical and analytical solutions for the fractional Riccati differential equation with delay term

$$D_*^\alpha y(x) = A(x) + B(x) y(\lambda x + \beta) + C(x) y^2(x),$$
$$x > 0, \ 0 < \alpha \le 1 \tag{1}$$

subject to the initial conditions

$$y(0) = \lambda, \tag{2}$$

where $A(x)$, $B(x)$, and $C(x)$ are given functions, α is a parameter describing the order of the fractional derivative and λ, β are appropriate constants, and $\lambda x + \beta > 0$ for all $x \in [0, 1]$. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha = 1$, the fractional equation reduces to the classical Riccati differential equation. The importance of this equation usually arises in the optimal control problems [10]. The existing literature on fractional differential equations tends to focus on particular values for the order α. In modern applications (see e.g., [11]) much more general values of the order appear in the equations, and therefore one needs to consider numerical and analytical methods to solve differential equations of arbitrary order. This equation is solved the numerically in [12–14].

We seek the approximate solution of (1) under the conditions (2) with the fractional Taylor series as $D_a^{k\alpha} y(x) \in C(a,b)$,

$$y_N(x) = \sum_{i=0}^{N} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_a^{i\alpha} y(x)\right)_{x=a}, \qquad (3)$$

where $0 < \alpha \le 1$. In recently, collocation method has become a very useful technique for solving equations [15–22]. This method transforms each part of the equation into matrix form and using the collocation points as

$$x_i = \frac{i}{N}, \quad i = 0,1,\ldots,N, \qquad (4)$$

we get the nonlinear algebraic equation. Then this equation is solved, we obtained the coefficients, the approximate solutions for various N. All computations are performed on the computer algebraic system Maple 13 in this paper.

2. Basic Definitions

In this section, we first give some basic definitions and then present properties of fractional calculus [1, 2, 23].

Definition 1. A real function $f(x)$, $x > 0$ is said to be in space C_μ, $\mu \in R \in$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in [0,\infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu$, $m \in N$.

Definition 2. The Riemann-Liouville fractional derivative of order α with respect to the variable t and with the starting point at $x = a$ is

$$_aD_x^\alpha f(x)$$

$$= \begin{cases} \dfrac{1}{\Gamma(-\alpha+m+1)} \dfrac{d^{m+1}}{dx^{m+1}} \\ \times \int_a^x (x-\tau)^{m-\alpha} f(\tau)\,d\tau, & 0 \le m \le \alpha < m+1 \\ \dfrac{d^m f(x)}{dt^m}, & \alpha = m+1 \in N. \end{cases} \qquad (5)$$

Definition 3. The Riemann-Liouville fractional integral of order α is

$$_aD_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)\,d\tau, \quad p > 0. \qquad (6)$$

Definition 4. The fractional derivative of $f(x)$ by means of Caputo sense is defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,d\tau \qquad (7)$$

for $n-1 < \alpha \le n, n \in N, x > 0, f \in C_{-1}^n$.

For the Caputo's derivative we have $D^\alpha C = 0$, C is a constant,

$$D^\alpha x^n = \begin{cases} 0, & n \in N, n < \lceil \alpha \rceil \\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & n \in N, n < \lfloor \alpha \rfloor. \end{cases} \qquad (8)$$

Theorem 5 (generalized Taylor formula). *Suppose that* $D_a^{k\alpha} f(x) \in C(a,b)$ *for* $k = 0,1,\ldots,n+1$, *where* $0 < \alpha \le 1$, *then one has [23]*

$$f(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_a^{i\alpha} f\right)(a)$$

$$+ \frac{\left(D_a^{(n+1)} f\right)(\xi)}{\Gamma((n+1)\alpha+1)} (x-a)^{(n+1)\alpha} \qquad (9)$$

with $a \le \xi \le x$, *for all* $x \in (a,b)$, *where*

$$D_a^{n\alpha} = D_a^\alpha \cdot D_a^\alpha \cdot D_a^\alpha \cdots D_a^\alpha \quad (n \text{ times}). \qquad (10)$$

3. Fundamental Relations

In this section, we consider the fractional Ricatti differential equations (1). We use the Taylor matrix method [15–22] to find the truncated Taylor series expansions of each term in expression at $x = c$ and their matrix representations for solving αth order linear fractional part and nonlinear part. We first consider the solution $y(x)$ of (1) defined by a truncated Taylor series (3). Then, we have the matrix form of the solution $y_N(x)$

$$[y_N(x)] = \mathbf{T}(x)\mathbf{A} = \mathbf{X}(x)\mathbf{M}_0\mathbf{A}, \qquad (11)$$

where

$$\mathbf{X}(x) = \begin{bmatrix} 1 & (x-c)^\alpha & (x-c)^{2\alpha} & \cdots & (x-c)^{N\alpha} \end{bmatrix},$$

$$\mathbf{M}_0 = \begin{bmatrix} \dfrac{1}{\Gamma(1)} & 0 & 0 & \cdots & 0 \\ 0 & \dfrac{1}{\Gamma(\alpha+1)} & 0 & \cdots & 0 \\ 0 & 0 & \dfrac{1}{\Gamma(2\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{1}{\Gamma(N\alpha+1)} \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} D_*^{0\alpha} y(c) \\ D_*^{1\alpha} y(c) \\ D_*^{2\alpha} y(c) \\ \vdots \\ D_*^{N\alpha} y(c) \end{bmatrix}. \qquad (12)$$

Then, the matrix representation of the function $D_*^\alpha y_N(x)$ becomes

$$D_*^\alpha y_N(x) = D_*^\alpha \mathbf{X}(x)\mathbf{M}_0\mathbf{A}, \tag{13}$$

$$
\begin{aligned}
D_*^\alpha \mathbf{X}(x) &= \left[D_*^\alpha 1 \quad D_*^\alpha (x-c)^\alpha \quad D_*^\alpha (x-c)^{2\alpha} \quad \cdots \quad D_*^\alpha (x-c)^{N\alpha} \right] \\
&= \left[0 \quad \frac{\Gamma(\alpha+1)}{\Gamma(1)} \quad \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}(x-c)^\alpha \quad \cdots \quad \frac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)}(x-c)^{(N-1)\alpha} \right] \\
&= \mathbf{X}(x)\mathbf{M}_1,
\end{aligned}
\tag{14}
$$

where

$$
\mathbf{M}_1 = \begin{bmatrix}
0 & \dfrac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & \cdots & 0 \\
0 & 0 & \dfrac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \dfrac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)} \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}.
\tag{15}
$$

Then, so the matrix representation of fractional differential part as

$$D_*^\alpha y_N(x) = \mathbf{X}(x)\mathbf{M}_1\mathbf{M}_0\mathbf{A}. \tag{16}$$

Additionally, using (11) we can write

$$\left[y_N(\lambda x + \beta) \right] = \mathbf{X}(\lambda x + \beta)\mathbf{M}_0\mathbf{A} = \overline{\mathbf{X}}(x)\mathbf{M}_0\mathbf{A}, \tag{17}$$

where

$$
\begin{aligned}
\overline{\mathbf{X}}(x) &= \mathbf{X}(\lambda x + \beta) \\
&= \left[1 \quad (\lambda x + \beta - c)^\alpha \quad \cdots \quad (\lambda x + \beta - c)^{N\alpha} \right].
\end{aligned}
\tag{18}
$$

Moreover, since [21, 22]

$$\mathbf{Y}^m = \mathbf{Y}^{m-1}\overline{\mathbf{Y}}, \tag{19}$$

where

$$
\mathbf{Y}^{m-1}(x) = \begin{bmatrix} y^{m-1}(x) \\ y^{m-1}(x) \\ \vdots \\ y^{m-1}(x) \end{bmatrix},
$$

$$
\overline{\mathbf{Y}}(x) = \begin{bmatrix}
y(x) & 0 & \cdots & 0 \\
0 & y(x) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & y(x)
\end{bmatrix},
\tag{20}
$$

where we compute the $D_*^\alpha \mathbf{X}(x)$, then

and using collocation points in (11)

$$\overline{\mathbf{Y}} = \overline{\mathbf{T}}\,\overline{\mathbf{A}}, \tag{21}$$

where

$$
\overline{\mathbf{T}}(x) = \begin{bmatrix}
\mathbf{T}(x) & 0 & \cdots & 0 \\
0 & \mathbf{T}(x) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mathbf{T}(x)
\end{bmatrix},
\tag{22}
$$

$$
\overline{\mathbf{A}} = \begin{bmatrix}
\mathbf{A} & 0 & \cdots & 0 \\
0 & \mathbf{A} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mathbf{A}
\end{bmatrix}
$$

then we construct the following relation

$$y(x_i)\,y(x_i) = y(x_i)\,y(x_i) = \left(\overline{\mathbf{T}}\,\overline{\mathbf{A}}\right)\mathbf{X}(x_i)\mathbf{M}_0\mathbf{A}. \tag{23}$$

Hence, the fundamental matrix relation of (1) is

$$
\begin{aligned}
\big(\mathbf{X}(x)\mathbf{M}_1\mathbf{M}_0 &- B(x)\overline{\mathbf{X}}(x)\mathbf{M}_0 \\
&- C(x)\left(\overline{\mathbf{T}}\,\overline{\mathbf{A}}\right)\mathbf{X}(x)\mathbf{M}_0 \big)\mathbf{A} = A(x).
\end{aligned}
\tag{24}
$$

Finally, we obtained matrix representation of the condition in (2)

$$\mathbf{U}_0 = \mathbf{X}(0)\mathbf{M}_0\mathbf{A} = \left[u_0 \ u_1 \ u_2 \ \cdots \ u_N \right] = [\lambda]. \tag{25}$$

4. Method of Solution

Using collocation points in (4), we can write (24)

$$
\begin{aligned}
\big(\mathbf{X}(x_i)\mathbf{M}_1\mathbf{M}_0 &- B(x_i)\overline{\mathbf{X}}(x_i)\mathbf{M}_0 \\
&- C(x_i)\left(\overline{\mathbf{T}}\,\overline{\mathbf{A}}\right)\mathbf{X}(x_i)\mathbf{M}_0 \big)\mathbf{A} = A(x_i)
\end{aligned}
\tag{26}
$$

or briefly the fundamental matrix equation

$$\left(\mathbf{X}\mathbf{M}_1\mathbf{M}_0 - \mathbf{B}\overline{\mathbf{X}}\mathbf{M}_0 - \mathbf{C}\left(\overline{\mathbf{T}}\,\overline{\mathbf{A}}\right)\mathbf{X}\mathbf{M}_0 \right)\mathbf{A} = \mathbf{F}, \tag{27}$$

where

$$\mathbf{X} = \begin{bmatrix} 1 & (x_0 - c)^\alpha & (x_0 - c)^{2\alpha} & \cdots & (x_0 - c)^{N\alpha} \\ 1 & (x_1 - c)^\alpha & (x_1 - c)^{2\alpha} & \cdots & (x_1 - c)^{N\alpha} \\ 1 & (x_2 - c)^\alpha & (x_2 - c)^{2\alpha} & \cdots & (x_2 - c)^{N\alpha} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (x_N - c)^\alpha & (x_N - c)^{2\alpha} & \cdots & (x_N - c)^{N\alpha} \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} A(x_0) \\ A(x_1) \\ A(x_2) \\ \vdots \\ A(x_N) \end{bmatrix},$$

$$\overline{\mathbf{X}} = \begin{bmatrix} 1 & (\lambda x_0 + \beta - c)^\alpha & (\lambda x_0 + \beta - c)^{2\alpha} & \cdots & (\lambda x_0 + \beta - c)^{N\alpha} \\ 1 & (\lambda x_1 + \beta - c)^\alpha & (\lambda x_1 + \beta - c)^{2\alpha} & \cdots & (\lambda x_1 + \beta - c)^{N\alpha} \\ 1 & (\lambda x_2 + \beta - c)^\alpha & (\lambda x_2 + \beta - c)^{2\alpha} & \cdots & (\lambda x_2 + \beta - c)^{N\alpha} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (\lambda x_N + \beta - c)^\alpha & (\lambda x_N + \beta - c)^{2\alpha} & \cdots & (\lambda x_N + \beta - c)^{N\alpha} \end{bmatrix}, \tag{28}$$

$$\overline{\mathbf{T}} = \begin{bmatrix} T(x_0) & 0 & 0 & \cdots & 0 \\ 0 & T(x_1) & 0 & \cdots & 0 \\ 0 & 0 & T(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & T(x_N) \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} B(x_0) & 0 & 0 & \cdots & 0 \\ 0 & B(x_1) & 0 & \cdots & 0 \\ 0 & 0 & B(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B(x_N) \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} C(x_0) & 0 & 0 & \cdots & 0 \\ 0 & C(x_1) & 0 & \cdots & 0 \\ 0 & 0 & C(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C(x_N) \end{bmatrix}.$$

Hence, the fundamental matrix equation (27) corresponding to (1) can be written in the form

$$\mathbf{WA} = \mathbf{F} \quad \text{or} \quad [\mathbf{W}; \mathbf{F}], \ \mathbf{W} = \begin{bmatrix} w_{i,j} \end{bmatrix}, \ i, j = 0, 1, \dots, N, \tag{29}$$

where

$$\mathbf{W} = \mathbf{XM}_1\mathbf{M}_0 - \mathbf{B}\overline{\mathbf{X}}\mathbf{M}_0 - \mathbf{C}\left(\overline{\mathbf{T}}\,\overline{\mathbf{A}}\right)\mathbf{XM}_0. \tag{30}$$

To obtain the solution of (1) under conditions (2), by replacing the row matrices (25) by the last 2 rows of the matrix (26), we have the new augmented matrix,

$$\left[\widetilde{\mathbf{W}}; \widetilde{\mathbf{F}}\right] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & A(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & A(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{N-20} & w_{N-21} & \cdots & w_{N-2N} & \vdots & A(x_{N-2}) \\ w_{N-10} & w_{N-11} & \cdots & w_{N-1N} & ; & A(x_{N-1}) \\ u_0 & u_1 & \cdots & u_N & ; & \lambda \end{bmatrix}. \tag{31}$$

So, we obtained a system of $(N + 1)$ nonlinear algebraic equations with unknown Taylor coefficients.

We can easily check the accuracy of the method. Since the truncated fractional Taylor series (3) is an approximate solution of (1), when the solution $y_N(x)$ and its derivatives

are substituted in (1), the resulting equation must be satisfied approximately; that is, for $x = x_q \in [0, 1]$, $q = 0, 1, 2, \dots$

$$E_N(x_q) = \left| D_*^\alpha y(x_q) - A(x_q) - B(x_q) y(\lambda x_q + \beta) \right. \\ \left. - C(x_q) y^2(x_q) \right| \cong 0. \tag{32}$$

5. Examples

In order to illustrate the effectiveness of the method proposed in this paper, several numerical examples are carried out in this section. In the following computations, for convenience, absolute errors between Nth-order approximate values y_N and the corresponding exact values y_{ex} as $N_e = |y_N - y_{ex}|$ are determined and all computations performed computer algebraic system with mathematical programing in Maple 13.

Example 1. Consider the following fractional Riccati equation:

$$D_*^\alpha y(x) = y^2(x) - x^2 y(x+1) + \frac{\Gamma(3)}{\Gamma(5/2)}x^{3/2} - 2x^3 - x^2,$$

$$x > 0, \ 0 < \alpha \leq 1 \tag{33}$$

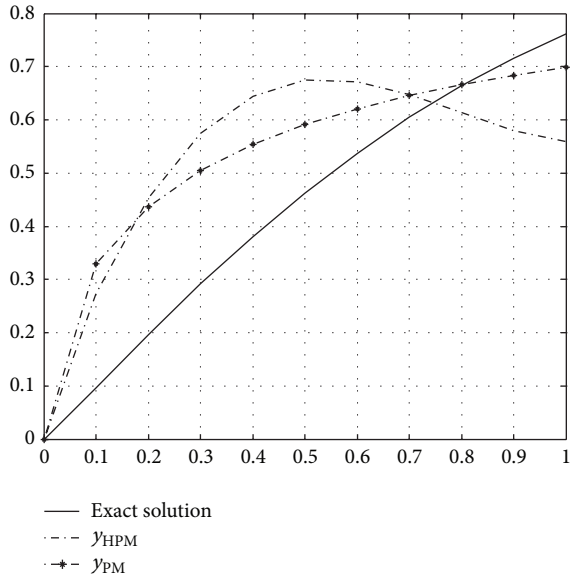with initial conditions

$$y(0) = 0. \tag{34}$$

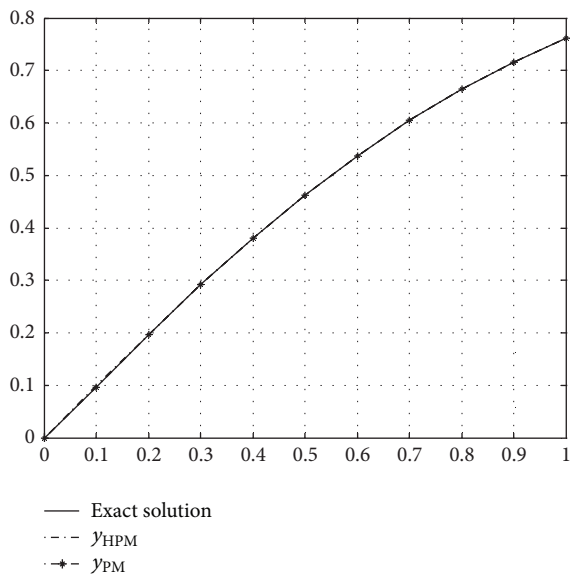FIGURE 1: Comparison of the HPM and PMN for $\alpha = 0.5$.

FIGURE 2: Comparison of the HPM and PM for $\alpha = 1$.

Then, $A(x) = \Gamma(3)x^{3/2}/\Gamma(5/2) - 2x^3 - x^2$, $B(x) = -x^2$, $C(x) = 1$. We assume that $\alpha = 1/2$, $0 \leq x \leq 1$, and we seek the approximate solutions y_N by Taylor series, for $c = 0$, $N = 4$

$$y_4(x) = \sum_{k=0}^{4} \frac{x^{k\alpha}}{\Gamma(k\alpha + 1)} \left(D_*^{k\alpha} y(x) \right)_{x=0} \tag{35}$$

with collocation points being

$$x_0 = 0, \qquad x_1 = \frac{1}{4},$$

$$x_2 = \frac{2}{4}, \qquad x_3 = \frac{3}{4}, \tag{36}$$

$$x_4 = 1.$$

Fundamental matrix relation of this problem is

$$\left(\mathbf{X}\mathbf{M}_1\mathbf{M}_0 - \mathbf{B}\overline{\mathbf{X}}\mathbf{M}_0 - \mathbf{C}\left(\overline{\mathbf{T}}\,\overline{\mathbf{A}} \right)\mathbf{X}\mathbf{M}_0 \right)\mathbf{A} = \mathbf{F}, \tag{37}$$

where

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & \dfrac{1}{\sqrt{4}} & \dfrac{1}{4} & \dfrac{\sqrt{4}}{16} & \dfrac{1}{16} \\ 1 & \dfrac{1}{\sqrt{2}} & \dfrac{1}{2} & \dfrac{\sqrt{2}}{4} & \dfrac{1}{4} \\ 1 & \dfrac{\sqrt{3}}{\sqrt{4}} & \dfrac{3}{4} & \dfrac{3\sqrt{3}}{4\sqrt{4}} & \dfrac{9}{16} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\overline{\mathbf{X}} = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 1 & \dfrac{1}{\sqrt{4}} + 1 & \dfrac{1}{4} + 1 & \dfrac{\sqrt{4}}{16} + 1 & \dfrac{1}{16} + 1 \\ 1 & \dfrac{1}{\sqrt{2}} + 1 & \dfrac{1}{2} + 1 & \dfrac{\sqrt{2}}{4} + 1 & \dfrac{1}{4} + 1 \\ 1 & \dfrac{\sqrt{3}}{\sqrt{4}} + 1 & \dfrac{3}{4} + 1 & \dfrac{3\sqrt{3}}{4\sqrt{4}} + 1 & \dfrac{9}{16} + 1 \\ 1 & 2 & 2 & 2 & 2 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0625 & 0 & 0 & 0 \\ 0 & 0 & 0.2500 & 0 & 0 \\ 0 & 0 & 0 & 0.5625 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{M}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{2}{\sqrt{\pi}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{4}{3\sqrt{\pi}} & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{2} \end{bmatrix},$$

$$\mathbf{M}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{2}{\sqrt{\pi}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{4}{3\sqrt{\pi}} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} 0.000000 \\ 0.188063 \\ 0.531923 \\ 0.977205 \\ 1.504505 \end{bmatrix}. \tag{38}$$

Also, we have the matrix representation of conditions,

$$y(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{A} = [0] \tag{39}$$

TABLE 1: Numerical result for Example 1.

x	Exact solution	Present Method					
		$N = 4$	$N_e = 4$	$N = 5$	$N_e = 5$	$N = 6$	$N_e = 6$
0.0	0.000	0.000000	$0.000E - 0$	0.000000	$0.000E - 0$	0.00000	$0.000E - 0$
0.2	0.040	0.039985	$0.150E - 4$	0.040000	$0.553E - 6$	0.040000	$0.740E - 8$
0.4	0.160	0.159993	$0.700E - 5$	0.160000	$0.352E - 6$	0.160000	$0.530E - 8$
0.6	0.360	0.359996	$0.446E - 5$	0.360000	$0.330E - 6$	0.360000	$0.497E - 8$
0.8	0.640	0.63996	$0.446E - 5$	0.640000	$0.337E - 6$	0.640000	$0.528E - 8$
1.0	1.000	0.999998	$0.219E - 5$	1.000000	$0.464E - 6$	1.000000	$0.673E - 8$

TABLE 2: The numerical results with comparison HP and PM ($N = 12$).

x	y_{exact}	y_{HPM}		y_{PM}	
		$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.096668	0.273875	0.099668	0.329929	0.096668
0.2	0.197375	0.454125	0.197375	0.436741	0.197375
0.3	0.291313	0.573932	0.291312	0.504823	0.291313
0.4	0.379949	0.644422	0.379944	0.553733	0.379949
0.5	0.462117	0.674137	0.462078	0.591156	0.462117
0.6	0.537050	0.671987	0.536857	0.620983	0.537050
0.7	0.604368	0.648003	0.603631	0.645461	0.604368
0.8	0.664037	0.613306	0.661706	0.665998	0.664037
0.9	0.716298	0.579641	0.709919	0.683545	0.716298
1.0	0.761594	0.558557	0.746032	0.698852	0.761594

and so we solve (37) and obtain the coefficients of the Taylor series

$$\mathbf{A} = \begin{bmatrix} 0 & -0.478771e - 4 & 0.189414e - 3 & -0.307615e - 3 & 2.000189 \end{bmatrix}. \tag{40}$$

Hence, for $N = 4$, the approximate solution of Example 1 is given as

$$y_5 = \frac{0.957543\sqrt{x}}{\sqrt{\pi}} + 0.189414e - 3x \\ - \frac{0.410153x^{3/2}}{\sqrt{\pi}} + 1.000094x^2. \tag{41}$$

Comparison of numerical results with the exact solution is shown in Table 1 for various N.

Example 2. Let us consider the following fractional Riccati equation [13]

$$D_*^\alpha y(x) = y^2(x) + 1, \quad 0 < \alpha \le 1 \tag{42}$$

subject to the initial condition

$$y(0) = 1. \tag{43}$$

Then, $A(x) = 1$, $B(x) = 0$, $C(x) = 1$. Fundamental matrix relation of this problem is

$$\left(\mathbf{XM}_1\mathbf{M}_0 - \mathbf{C}\left(\overline{\mathbf{T}}\,\overline{\mathbf{A}}\right)\mathbf{XM}_0\right)\mathbf{A} = \mathbf{F}. \tag{44}$$

Also, we have the matrix representation of conditions,

$$y(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{A} = [0]. \tag{45}$$

The exact solution, when $\alpha = 1$, is

$$y(x) = \frac{e^{2x} - 1}{e^{2x} + 1}. \tag{46}$$

We approximately solve the fractional Ricatti equation for $N = 12$ and obtained the approximate solution for $\alpha = 1$,

$$
\begin{aligned}
y_{12} = \, & x - 0.559171e - 6x^2 - 0.333319x^3 - 0.155476e \\
& - 3x^4 + 0.134388x^5 - 0.467422e - 2x^6 - 0.398472e \\
& - 1x^7 - 0.295050e - 1x^8 + 0.640148e \\
& - 1x^9 - 0.390402e - 1x^{10} + 0.109027x^{11} \\
& - 0.116904e - 2x^{12}.
\end{aligned}
$$

(47)

Table 2 shows the approximate solutions for Example 2 obtained by the present method and compare with homotopy perturbation method for different α. From the numerical results in Table 2, present method is in high agreement with the exact solution than homotopy perturbation method [13]. Moreover, using the numerical result in Table 2, Figures 1 and 2 are plotted. Figure 2 shows that numerical solution y_{PM} is so closed to the exact solution.

6. Conclusion

In real world systems, delays can be recognised everywhere and there has been widespread interest in the study of delay differential equations for many years. Although it seems natural to model certain processes and systems in engineering and other sciences with this kind of equation, only in the last few years has the attention of the scientific community been devoted to them.

In this study, we present a Taylor collocation method for the numerical solutions of fractional Riccati differential equation with delay term. This method transforms fractional Riccati differential equation with delay term into matrix equations. The desired approximate solutions can be determined by solving the resulting system, which can be effectively computed using symbolic computing codes on Maple 13. Examples show that Taylor collocation method has been successfully applied to find the approximate solutions of the fractional Riccati differential equation. Graphics and tables show that this method is extremely effective and practical for this sort of approximate solutions.

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