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METHODOLOGIES AND APPLICATION

Discrete Ricci curvature-based statistics for soft sets

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Abstract
Soft sets are efficient mathematical structures to model systems in multiple relations. Since a soft set is basically set system, it is possible to endow them with a proper distance function to obtain a metric space. By this embedding, we propose a discretization of the Ricci curvatures that stresses the relational character of universe elements in a soft set through the analysis of parameters rather than the elements themselves. The Forman and Ollivier-type Ricci curvatures we propose here quantifies the trade-off between parameter size and the cardinality of participation of parameterized universe elements in other parameters. Such discretizations of the Ricci curvature have already been applied to complex systems; however, it has not yet been formulated for soft sets. In this study, our main question is whether the defined geometric concept determines statistics for soft set models. Two examples are discussed for the answer to this question. The first example Ricci on soft sets model of occupational accidents occurred in Turkey in 2013–2014 is compared with the Wasserstein distance of the curvature distributions. The second example is the use of Ricci curvatures as an indicator in the soft sets model of a financial system while the system is in stress. These real world examples show that discrete Ricci curvatures for soft sets offer effective statistics.

Keywords Soft sets · Computational simplex · Forman Ricci curvature · Ollivier Ricci curvature

1 Introduction

The complexities of a multi-agent system emerge from the interaction of many components. The problem of determining the characteristics of systems such as the human brain and world economy, whose behavior is difficult to predict and control, is one of the fundamental questions of multi-agent systems. Uncertainty can also be observed in the behavior of components in nonlinear relationships. In order to define such uncertainties, soft set theory emerges as an effective tool. Soft set theory is firstly presented by Molodtsov (1999); such as the theory is separated by arbitrary selection of parameters regarding to fuzzy sets, vogue sets, and rough sets theories. The main characteristic of soft sets is that they are completely free from the membership degrees. Mathematically, a soft set is characterized by the help of arbitrary parameter transformation of the elements given in the initial universe.

One may conclude that a soft set is a neighborhood system which are special case of context-dependent fuzzy sets (Aktaş and Cağman 2007). To be neighborhood system, identity leads at least one topological structure on a soft set, and context dependency identity makes soft sets to applicable in many soft computing areas. Intelligence computations and missing value predictions play key role in soft computing (Al-Janabi and Alkaim 2020; Al-Janabi et al. 2019, 2014; Ali 2013; Kalajdzic et al. 2015; Alkaim and Al Janabi 2019; Patel et al. 2015). Soft sets are also used to perform these types of tasks. For instance, recently, Alcantud and Santos-Garcia (Alcantud and Santos-García 2016, 2017) have contributed to decision making with incomplete information. Particularly, in Alcantud and Santos-García (2016), authors show that soft sets are efficient mathematical structures to perform decision making in Economics. Their method is based on defining Laplacian for soft sets. Similarly, recent surveys (Ma et al. 2017; Zhan and Zhu 2015; Zhan et al. 2017) show that soft sets can be used for decision making in multiple disciplines. One of the remarkable applications of soft sets emerges in conflict analysis. In Sutoyo et al. (2016), authors briefly show that binary relations of coalition, neutrality, and conflict among agents can be efficiently modeled via soft

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sets. Besides, there are some recent studies show the applications of soft sets in decision-making practices in medicine. In medical diagnosis, intuitionistic and interval-valued fuzzy soft sets are used (Chetia and Das 2010; Saikia et al. 2003) as fuzzy techniques. Moreover, Yuksel et al. (2013) use soft set theory in the diagnosis of risk of prostate cancer, and Alcuntad et al. extend this idea to lung cancer resections and surgical decision making in Alcantud et al. (2019). Soft sets, which are very effective in decision supper, are used not only in these processes, but also for modeling systems with multiple interactions. For instance, in Balci and Akguller (2015), authors briefly introduce soft set model of a metabolic system and adapt mathematical morphological operations. Furthermore, in Balci and Akguller (2016), authors present a method to obtain soft set models of financial systems and analyze systems in econophysics point of view.

The topological structures of soft sets, which are considered as neighborhood systems, are defined by the interactions of the elements through parameters in the system. Interaction through these parameters indicates that the soft set theory is an effective tool to study the structural feature of systems with uncertainty conditions. Although a soft set can be considered as a set-valued mapping, we need to conduct a more detailed research to examine the effect of parameterization on structural analysis. Hence, our focus in this study is to develop quantitative understanding of the interactions in multi-agent systems modeled by soft sets. Such numerical understanding is performed on the trade-off between parameter volumes and the cardinality of participation of parameterized universe elements in other parameters. Since soft sets encode a strong information system in themselves, it is possible to consider them as abstract manifolds. Such manifolds have geometrization in some n-dimensional space. The concept of geometric soft sets is first presented in Akguller (2017), as the initial universe is the points in general position in $\mathbb{R}^d$ and the parameters are determined by incidence mapping. Different than the other well-known computational complexes, the geometric soft sets do not have to have heredity property. Such identity of geometric soft sets lets us to determine fuzziness in the computational complex. Geometrization of soft sets let us to determine soft set statistics by using Ricci curvature which is one of the fundamental concepts in Riemannian geometry. Let us assume M is a complete connected Riemannian manifold equipped with the metric g. Then, Ricci curvature tensor measures the degree to which the geometry determined by g differs from that of Euclidean space (Jost and Jost 2008). The Ricci curvature in soft set setting can be consider in two ways: first one captures the volume growth of parameters, and the second one uses transportation distance between topological balls emerge from parametrization. Both approaches are consistent with the infinitesimal setting definition of Ricci curvature that is quantifying divergence of geodesics and volume growth. There are also several studies to define the different types of Ricci curvatures in more general metric spaces (Erbar et al. 2015; Fathi and Maas 2016; Lott and Villani 2009; Ni et al. 2015; Ollivier 2007; Saucan et al. 2019).

In this present study, we employ two different approaches to define Ricci curvature on soft sets. Our first approach is based on a definition proposed by Forman (2003). Forman's definition on discrete Ricci curvature is based on Bochner-Weitzenbock decomposition of the Laplacian. Such discretization is recently applied to network science studies (Ache and Warren 2019; Gao et al. 2019; Ni et al. 2019; Saucan and Weber 2018). Second approach follows general framework of finite Markov processes. In Ollivier (2007), Ollivier presented that discrete Ricci curvature of a metric measure space can be defined by associating a probability measure on a point. It should be noted that Ricci curvature controls the local behavior of geodesics. In the neighborhoods with negative curvature, the geodesics diverge, whereas when the curvature is positive, they converge. Ricci curvature is a fundamental tool also in discrete heat calculus by providing an upper bound on the heat kernel (Münch and Wojciechowski 2019; Wang et al. 2014). However, in this study, our interest in discrete Ricci curvatures such as Forman and Ollivier types rather stems from its discrete heat calculus properties in terms of volumes of the parameters and transportation cost.

In the subsequent sections, first, we give some basic definitions on geometric soft sets. In order to define Forman-Ricci curvature, we present a soft set Laplacian defined on p-chains. Then, we define Forman-Ricci curvature on soft sets which have weighted parameters. Afterward, we extend this idea to general soft sets. Similarly, in order to define Ollivier-Ricci curvature, we first give a probability measure on parameters. Then, we present Ollivier-Ricci curvature on soft sets by using Wasserstein-1 distance. This latter definition of Ricci curvature depends on the solution of multi-marginal optimization problem. Therefore, we use Wasserstein barycenter solution of such problem in order to avoid computational complexity. In Sect. 3, we give computational results on the Forman and Ollivier-type Ricci curvatures. We apply these notions on some real multi-agent systems such as the occupational accidents happened in Turkey during 2013–2014, and stock market crisis of 2008. The details of soft set representations of these systems are given in details. Our results indicate that such soft set statistics is useful to determine similar class of a system by distributions of discrete Ricci curvatures. Furthermore, such statistics can also be used as an indicator of a stock market crisis. Furthermore, in Sect. 4, we give detailed discussion on this present method and obtained results. Finally, in Sect. 5, we give concluding remarks and mention some further studies. It is sincerely hoped that this study can shed light on the development of further researches on geometry of soft sets.

2 Discrete Ricci curvatures for soft sets

In this section, we present several definitions on geometric soft sets including rough Laplacian, Forman and Ollivier type of discretization of Ricci curvatures on soft sets.

2.1 Geometric soft sets and Laplacian

In mathematical point of view, a soft set (F, E) is a parameterized family of subsets of the universe set U which can be stated as a set of ordered pairs

$$(F, E) = \{(e, F(e)) \,:\, e \in E, F(e) \subset U\},$$

where $F : E \to 2^U$ is a parameter mapping (Molodtsov 1999). The basic operations on soft set can be found in Maji et al. (2003), and soft and fuzzy-soft topological identities can be found in Hazra et al. (2012); Varol and Aygun (2012).

The definition of the geometric soft sets regarding to incidence relation is first given in Akguller (2017) by considering the elements of the universe U are the points in $\mathbb{R}^d$ in general position.

Definition 1 Let $U \subset \mathbb{R}^d$ be the finite set of points in general position, $A \subseteq U$, and $P(A, i)$ denotes the set of subsets of A with i elements. For $F_A : E \to 2^A \setminus \{\emptyset\}$ incidence mapping, (F_A, E) is called a geometric soft set if

i. for $A = \{a_1, \ldots, a_k\}$, the tuple $(e_0, P(A, 1)) \in (F_a, E)$
ii. for all $i = 1, \ldots, k-2$, if $(e_{k-1}, P(A, k)) \in (F_a, E)$, then $(e_{i-1}, P(B, i)) \in (F_A, E)$ for $\exists B \subset A$.

The soft p-face of a geometric soft set (F_A, E) is a parametrization of cardinality $p+1$, and $S_p((F_A, E))$ denotes the set of all soft p-faces of a (F_A, E). The soft faces that are maximal under soft inclusion are called soft facets. A geometric soft set (F_A, E) is said to be regular if all facets have the same dimension.

In order to explain the geometric realization of soft sets and corresponding concepts, let us consider the geometric soft set

$$(F_A, E) = (F_{A_1}, E) \cup (F_{A_2}, E) \cup (F_{A_3}, E), \tag{1}$$

where

$$(F_{A_1}, E) = \left\{ \begin{array}{l} (e_0, \{\{a\}, \{b\}, \{c\}, \{d\}\}) \\ (e_1, \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}) \\ (e_2, \{\{a, b, c\}, \{a, c, d\}, \{a, b, d\}\}) \\ (e_3, \{\{a, b, c, d\}\}) \end{array} \right\},$$

$$(F_{A_2}, E) = \left\{ \begin{array}{l} (e_0, \{\{b\}, \{d\}, \{e\}\}) \\ (e_1, \{\{b, d\}, \{d, e\}\}) \\ (e_2, \{\{b, d, e\}\}) \end{array} \right\},$$

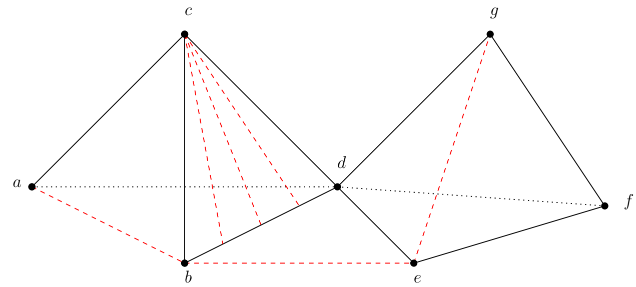

Fig. 1 Geometric realization of (F_A, E) defined as in 1

$$(F_{A_3}, E) = \left\{ \begin{array}{l} (e_0, \{\{d\}, \{e\}, \{f\}, \{g\}\}) \\ (e_1, \{\{d, e\}, \{e, f\}, \{f, g\}, \{d, g\}\}) \\ (e_2, \{\{d, e, f\}, \{d, e, g\}, \{d, f, g\}, \{e, f, g\}\}) \end{array} \right\}.$$

Geometric realization of (F_A, E) is given in Fig. 1. In this realization, the elements of the sub-soft sets without heredity are presented with dashed-red lines. We shall note that, the volume element of (F_{A_1}, E) is included whilst (F_{A_3}, E) excluded. (F_A, E) given in Equation 1 is not regular and has dimension 3. 2-faces of (F_{A_1}, E) are $(e_2, \{\{a, b, c\}\}$,$(e_2, \{\{a, c, d\}\}$ and $(e_2, \{\{a, b, d\}\})$. Moreover, 2-chain on (F_{A_1}, E) can be obtained by

$$(e_2, \{\{a, b, c\}\}) \widetilde{\oplus} (e_2, \{\{a, c, d\}\}) \widetilde{\oplus} (e_2, \{\{a, b, d\}\}),$$

on free Abelian soft group $(\widetilde{G}, \widetilde{\oplus})$. To our best knowledge, algebraic identities of such group have not been studied yet. Since it is subject to another study, we give no further details on $(\widetilde{G}, \widetilde{\oplus})$.

Forman-type discretization of Ricci curvature on soft sets is given regarding to soft set Laplacian that we define in this study. The definition of Laplacian operator for soft sets in previous studies are defined by the parameterization numbers of the elements of the initial universe. However, such definition would be insufficient to obtain geometric statistics. Moreover, only examining the cardinality of the parameters will be insufficient in determining the trade-off between parameters of soft sets. The Laplacian operator we present in this study is obtained from the adjoint of p-co-chains of geometric soft sets. Moreover, such Laplacian can be useful for spectral analysis in soft set theoretical point of view.

Definition 2 Let (F_A, E) be a geometric soft set with nonempty parameter and universe set. The dual of the soft set (F_A, E) is defined with $F^* : A^* \subset U \to E$, where U is the universe set and E is the parameter set of (F_A, E) and denoted by (F_A^*, E^*).

The p-th chain group $C_p((F_A, E), \widetilde{G})$ of (F_A, E) with coefficients in $\widetilde{G}$ is a vector space over the real field $\mathbb{R}$ with basis $B_p((F_A, E), \widetilde{G})$. Besides, the p-th co-chain group $C^p((F_A, E), \widetilde{G})$ is defined as the dual of the p-th chain group.

Definition 3 For the co-chain groups, the geometric soft set boundary maps $\delta_p : C^p((F_A, E), \widetilde{G}) \to C^{p+1}((F_A, E), \widetilde{G})$, $p \geq -1$, are defined by

$$(\delta_p f)\Big[(e_0, P(A,1)), \ldots, (e_p, P(A, p+1))\Big] = \sum_{j=0}^{p+1} (-1)^j f\Big[(e_0, P(A,1)), \ldots, \overline{(e_j, P(A, j+1))}, \ldots, (e_p, P(A, p+1))\Big] \quad (2)$$

for $f \in C^p((F_A, E), \widetilde{G})$, where $\overline{(e_j, P(A, j+1))}$ means that the soft element

$(e_j, P(A, j+1))$

is removed.

Definition 4 An inner product on the space $C^p((F_A, E), \widetilde{G})$ is defined by

$$< f, g >_{C^p} = \sum_{\mathcal{F} \in S_p((F_A, E))} \overline{\omega}(\mathcal{F}) f(\mathcal{F}) g(\mathcal{F}) \quad (3)$$

for $f, g \in C^p((F_A, E), \widetilde{G})$. $\overline{\omega} : \bigcup_{p=0} S_p((F_A, E)) \to \mathbb{R}^+$ is called the weight function of the inner product.

Definition 5 The adjoint operator $\delta_p^* : C^p((F_A, E), \widetilde{G}) \to C^{p+1}((F_A, E), \widetilde{G})$ of δ_p is defined by

$$< \delta_p f, g >_{C^{p+1}} = < f, \delta_p^* g >_{C^p}, \quad (4)$$

where $f \in C^p((F_A, E), \widetilde{G})$ and $g \in C^{p+1}((F_A, E), \widetilde{G})$.

Definition 6 The p-dimensional soft set Laplacian $\square_p : (F_A, E_p) \to (F_A, E_p)$ is defined by

$$\square_p = \delta_p^* \delta_p + \delta_{p-1} \delta_{p-1}^*. \quad (5)$$

2.2 Forman-Ricci curvature for soft sets

Such Laplacian defined in Definition 6 regarding to adjoint operator leads us to define Forman-type Ricci curvature for soft sets. In combinatorial approach, the canonical decomposition of Eq. 5 yields us the curvature function. Before giving further information about the computation of Forman-Ricci curvature for soft sets, we need to present a definition for the soft sets statistics.

Definition 7 The function $\omega : F_A \to \mathbb{R}^+$ defined of the parameter set of the soft set (F_A, E) is called the weight function. A weighted soft set then represented by the triple (F_A, E, ω).

A weight defined on the parameter can be considered as the measure of how strongly the elements of the set A are parameterized. The idea of giving such definition is actually based on the definition of Forman-Ricci curvature on soft sets, because the definition of $\mathbb{F}_p$ does not depend on the weights and makes the Forman-Ricci curvature extremely versatile.

Now, let us denote $(e_{p-1}, P(\{a_1, \ldots, a_p\}, p)) \in (F_A, E, \omega)$ with $F(e^p)$. Then, we are able to define the Forman-Ricci curvature for $F(e^p)$ with

$$\mathbb{F}_p = \mathbb{F}(F(e^p)) = \omega(F(e^p)) \Bigg[\Bigg(\sum_{F(e^p) \subset F(a^{p+1})} \frac{\omega(F(e^p))}{\omega(F(a^{p+1}))} + \sum_{F(b^{p-1}) \subset F(e^p)} \frac{\omega(F(b^{p-1}))}{\omega(F(a^p))} \Bigg) - \sum_{F(c^p) \bowtie F(e^p)} \Bigg| \sum_{F(c^p) \subset F(a^{p+1})} \frac{\sqrt{\omega(F(e^p))\omega(F(c^p))}}{\omega(F(a^{p+1}))} - \sum_{F(b^{p-1}) \subset F(c^p)} \frac{\sqrt{\omega(F(b^{p-1}))}}{\omega(F(e^p))\omega(F(c^p))} \Bigg| \Bigg], \quad (6)$$

where $\subset$ is crisp set inclusion operator, and the relation $F(e^p) \bowtie F(c^p)$ is defined as there exists $p+1$ dimensional $(e, F(e))$ such that $F(e^p)$ and $F(c^p)$ are both subsets of $(e, F(e))$ or $p-1$ dimensional $(e, F(e))$ such that $F(e^p)$ and $F(c^p)$ are both includes $(e, F(e))$. We call the $\bowtie$ relation as soft paralleling relation.

The Forman-Ricci curvature defined in Equation 6 is presented on a geometric soft set. However, in the real-world application, such restriction on the geometry of a soft set may not be applicable. Hence, by considering the volumes of the parameters as their cardinality, we may extend such definition to general soft sets.

Definition 8 Let (F_A, E) be a soft set. The neighborhood of $a_i \in A$ on (F_A, E) is the set $\mathcal{N}(a_i) = \bigcup_j N_j$, where $N_j = \{a_k : a_i \in F(e_j) \text{ and } a_k \in F(e_j)\}$.

Definition 9 Let (F_A, E) be a soft set. For $a_i \in A$, the number of parameters assigned to a_i is called the soft degree of a_i and denoted by $\bar{d}_{a_i}$. Similarly, for $e_j \in E$, the cardinality of $F(e_j)$ is called the soft degree of e_j and denoted by $\bar{d}_{e_j}$.

If $\bar{d}_{e_j} = 2$ for all $j = 1, \ldots, m$, that is $F(e_j) = \{a_1^j, a_2^j\}$, then the Equation 6 reduces to be on 2-regular weighted geometric soft set with

$$\mathbb{F}(F(e)) = 2 - \omega(F(e)) \sum_{a_i \in \mathcal{N}(a_1^j)} \frac{1}{\sqrt{\omega(F(e))\omega(F(e_j))}} - \omega(F(e)) \sum_{a_i \in \mathcal{N}(a_2^j)} \frac{1}{\sqrt{\omega(F(e))\omega(F(e_j))}} \quad (7)$$

Furthermore, if we separate the contributions of the element of A and assume (F_A, E, ω) does not have to be regular soft set, then it is possible to extend the Equation 7 to the general soft sets with

$$\mathbb{F}(F(e)) = \omega(F(e))\left(\sum_{a_i\in F(e)}\left(\frac{1}{\omega(F(e))} - \sum_{a_i\in F(e_j)}\frac{1}{\sqrt{\omega(F(e))\omega(F(e_j))}}\right)\right). \tag{8}$$

For unweighted case of soft set (F_A, E), Equation 8 simplifies to

$$\mathbb{F}(F(e)) = \sum_{a_i\in F(e)}\left(2-\bar{d}_{a_i}\right) = 2\bar{d}_{e_i} - \sum_{a_i\in F(e)}\bar{d}_{a_i}, \tag{9}$$

which is bounded below by $\bar{d}_{e_i}(2-|E|$ when $\bar{d}_{a_i} = |E|$ for every $a_i \in F(e)$, and bounded above by 1 when $\sum_{a_i\in F(e)}\bar{d}_{a_i} = \bar{d}_{e_i}$. In other words, the minimum curvature occurs when every element in $F(e)$ belongs to each parameters, the maximum is attained for an empty parameter.

2.3 Ollivier-Ricci curvature for soft sets

In this subsection, we introduce Ollivier-type Ricci curvature discretization for soft sets. In this sense, we first define probability measures on parameters and initial universe, and then Ollivier-Ricci curvature is defined as the optimal transportation problem.

In order to present Ollivier type of Ricci curvature discretization on geometric soft sets, first consider following definitions on the topological structure of geometric soft sets. For any two parameters $F(e)$ and $F(e')$ with $\bar{d}_e = \bar{d}_{e'} = p$. $F(e)$ and $F(e')$ are said to be connected if $\partial F(e)\cap\partial F(e') \neq \emptyset$ and denoted by $F(e)\sim F(e')$. $\partial F(e)$ denotes the boundary of a parameter mapping $F(e)$. A soft path from one $F(e)$ to other $F(e')$ is a sequence of connected parameters $\left\{F(e_j)\sim F(e_{j+1})\right\}_{j=0}^{j=n}$, where $F(e_0) = F(e)$ and $F(e_n) = F(e')$. A geometric soft set is said to be connected if any two soft elements $(e_p, (A, p+1))$ and $(e'_p, (A, p+1))$ can be connected by a soft path.

In order to define Ricci curvature, we define the Wasserstein-1 distance between probability measures on geometric soft sets.

Definition 10 The Wasserstein-1 distance between any two probability measures μ_1 and μ_2 on A of (F_A, E) is given by

$$W_1(\mu_1,\mu_2) = \inf_{\pi}\sum_{a_1,a_2\in A}\pi(a_1,a_2)d(a_1,a_2), \tag{10}$$

where the coupling $\pi : A\times A \to [0,1]$ runs over all maps satisfying

$$\sum_{a_1\in A}\pi(a_1,a_2) = \mu_1(a_1),\ \sum_{a_2\in A}\pi(a_1,a_2) = \mu_2(a_2), \tag{11}$$

and $d(a_1, a_2)$ is the minimum of the lengths of the soft paths from $F(e^{a_1})$ to $F(e^{a_2})$ such that $a_i \in F(e^{a_i})$ for $i = 1, 2$.

In the transportation distance between topological balls approach, we may follow up two different probability measures on soft sets. First measure is defined between parameters, and the second measure is defined on a parameter itself.

Now, let us define the first probability measure on $S_p((F_A, E))$.

Definition 11 For any $\varepsilon \in [0, 1]$,

$$\mathbb{P}^{\varepsilon}_{F(e)}(F(e')) = \begin{cases} 1-\varepsilon+\dfrac{\varepsilon}{p+1}\displaystyle\sum_{F(\bar{e})\in\partial F(e)}\dfrac{\omega(F(e))}{\bar{d}_{\bar{e}}}, & F(e)=F(e'),\\ \varepsilon\dfrac{\omega(F(e'))}{(p+1)\bar{d}_{\bar{e}}}, & \partial F(e)\cap\partial F(e') = F(\bar{e}),\\ 0, & \text{otherwise}\end{cases} \tag{12}$$

is a probability measure on $S_p((F_A, E))$.

By using the Wasserstein distance defined in Equation 10, we may define discrete Ricci curvature regarding to the probability measure given in Definition 11.

Definition 12 For any $\varepsilon \in [0, 1]$ and for any two distinct $F(e)$ and $F(e')$, the ε-Ricci curvature of $F(e)$ and $F(e')$ is defined by

$$\mathbb{O}_{\varepsilon}(F(e), F(e')) = 1 - \frac{W(\mathbb{P}^{\varepsilon}_{F(e)}, \mathbb{P}^{\varepsilon}_{F(e')})}{d_H(F(e), F(e'))}, \tag{13}$$

where d_H is the crisp Hausdorff distance.

Definition 13 For any two distinct $F(e)$ and $F(e')$, the Ollivier-Ricci curvature is defined by

$$\mathbb{O}(F(e), F(e')) = \lim_{\varepsilon \to 0} \frac{\mathbb{O}_\varepsilon(F(e), F(e'))}{\varepsilon}. \tag{14}$$

Such discretization of Ricci curvature regarding to Wasserstein distance depends on two distinct parameters. However, our goal is to capture geometry on each parameters. Therefore, we present another discretization regarding to multi-marginal optimal transport problem.

Definition 14 Let (F_A, E) be a geometric soft set. The sequence

$$\sigma_k = a_1, F(e_1), a_2, F(e_2), \ldots, F(e_{k-1}), a_k \tag{15}$$

is called a soft connection sequence between the elements $a_1, a_k \in A$. Besides, if interior elements of σ_k are chosen randomly, then σ_k is called a random soft connection sequence.

Definition 15 The uniform random soft sequence initialized at $a_i \in A$ has a probability measure $\mathbb{P}^{\sigma_i}$ with

$$\mathbb{P}^{\sigma_i}(a_j) = \sum_{a_i, a_j \in F(e)} \frac{1}{\bar{d}_{a_i}} \frac{1}{\bar{d}_e - 1}. \tag{16}$$

Definition 16 For a geometric soft set (F_A, E), the Ricci curvature of a parameter $F(e)$ is defined as

$$\mathbb{O}_F(F(e)) = 1 - \frac{W(F(e))}{|A| - 1}, \tag{17}$$

where $W(F(e))$ is the minimum of the multi-marginal optimal transport problem

$$W_1(F(e)) = \min_{\pi \in \Pi(\mathbb{P}^{\sigma_1}, \ldots, \mathbb{P}^{\sigma_n})} \sum_{\mathbf{a^n} \in \mathbf{A^n}} c(\mathbf{a}^n)\pi(\mathbf{a}^n) \tag{18}$$

with $\mathbf{a}^n = (a_1, \ldots, a_n)$ and $c(a_1, \ldots, a_n) = \min_{b \in A} \sum_{i=1}^{n} d(a_i, b)$.

The solution of such multi-marginal optimal transport problem is a linear program. However, its computational complexity grows exponentially. Therefore, we need to employ the barycenters to solve such problem efficiently. In order to determine barycenters of soft sets, we first need to remember the Definition 2. One may concludes by the definition that the duals of two isomorphic geometric soft sets are also isomorphic to each other. By introducing (F_A^*, E^*), we are able to determine Wasserstein barycenters with

$$bc(F(e)) = \inf_{\pi \in \Pi(\mathbb{P}^{\sigma_1}, \ldots, \mathbb{P}^{\sigma_n})} \sum_{i=1}^{n} W_1(\pi_i, \pi). \tag{19}$$

Hence, the Ricci curvature of a parametrization $F(e)$ can be computed by

$$\mathbb{O}_F(F(e)) = 1 - \frac{bc(F(e))}{|A| - 1}. \tag{20}$$

We need to remark that such barycenter definition is based on Wasserstein distance that is $\mathbb{O}_F(F(e))$ can be computed on any soft set. If we restrict our idea to A to be embedded in $\mathbb{R}^n$, the barycenters can be computed regarding to Euclidean distance.

3 Applications

In this section, we consider two examples of soft sets and present the computational results on discrete Ricci curvatures of the parameters. In order to state such soft set statistics, first example is chosen to be steady soft sets, and the second example is chosen to be a time varying soft set.

3.1 Occupational accidents

The first one is the soft set representations of the occupational accidents data of Turkey that happened in the period of 2013–2014. We need to denote that these soft sets are not the geometric ones. 10000 of the data were selected and 18 of the sectors with the most occupational accidents were taken into consideration. The NACE codes and their labels are presented in Table 2 in Appendix I. We refer readers (statbank.cso.ie/px/u/NACECoder/) for synonyms and more details on the codes. According to the six NACE code of the sectors examined, a total of 18 different soft sets are obtained by taking the quartet NACE code of the sectors in close relation with each other, the universe of work accidents, the parameters of information in the work accidents and the parameters. Each soft set is shown as a quad NACE code $(F_{\mathrm{NACE}}, A_{\mathrm{NACE}})$. The inputs of work accident information are taken as Number of Working Days, Age, Gender, Marital Status, Work Day Loss, Vocational Training, Occupational Safety Education, Educational Status, Number of Persons in the Accident. A set of parameters was taken in subsets to select work accident information as the main title and then 34 parameters are obtained. The more details on data and the parameters can be found in Balci and Tuna (2018). In Table, 1 we present the parameters.

Since $(F_{\mathrm{NACE}}, A_{\mathrm{NACE}})$ is un-weighted, it is possible to determine the Forman-Ricci curvatures of the parameter sets by using Equation 9 directly. Similarly, the Ollivier-Ricci curvature $\mathbb{O}_F(F_{\mathrm{NACE}}(e))$ is computed by using Eq. 20 with assuming each parameter has constant weight of 1.

The distributions of both discrete Ricci curvatures for $(F_{\mathrm{NACE}}, A_{\mathrm{NACE}})$ are presented in Figs. 2 and 3.

Table 1 The parameters list for constructing $(F_{\text{NACE}}, A_{\text{NACE}})$ Balci and Tuna (2018)

Number of working days ($t \equiv$ days)					
$0 \le t < 400$	$400 \le t < 1000$	$1000 \le t < 2000$	$2000 \le t < 3000$	$3000 \le t < 4000$	$t \ge 400$
Age ($t \equiv$ years)					
$18 \le t < 25$	$25 \le t < 30$	$30 \le t < 35$	$35 \le t < 40$	$40 \le t < 45$	$t \ge 45$
Working days loss ($t \equiv$ days)					
$0 \le t \le 1$	$1 < t \le 3$	$3 \le t < 5$	$5 \le t < 8$	$8 \le t < 10$	$t \ge 10$
Number of personsin the accident					
1		1-3		>3	
Educational status					
Elementary School		Secondary School		High School	University/Graduate
Gender					
Male			Female		
Martial status					
Married		Bachelor/Bachelorette		Other	
Vocational training					
Yes			No		
Occupational safety education					
Yes			No		

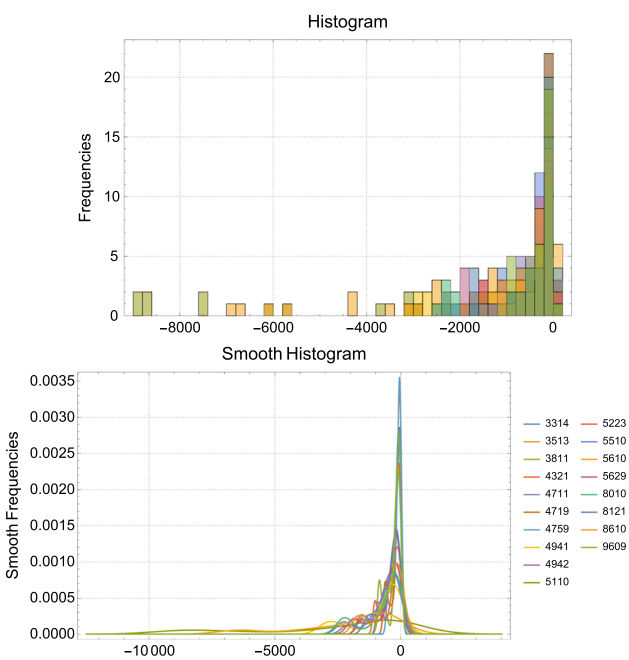

Fig. 2 Histograms of Forman-type discretization of Ricci curvature for $(F_{\text{NACE}}, A_{\text{NACE}})$

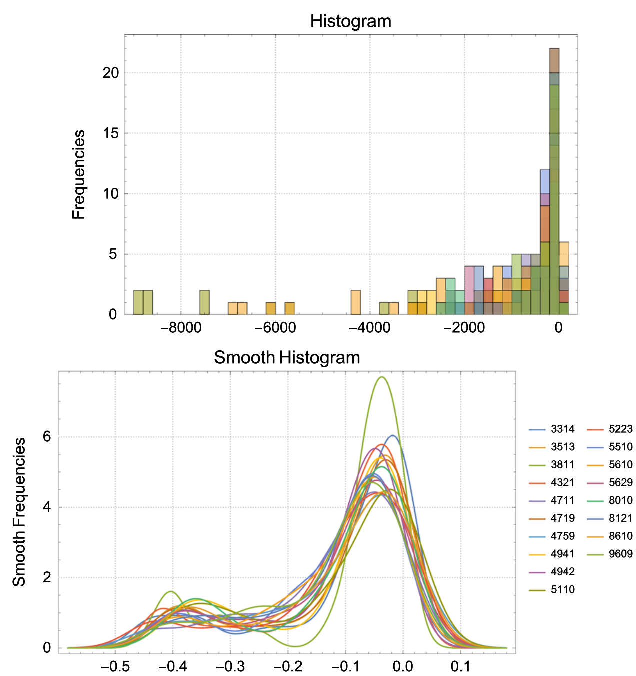

Fig. 3 Histograms of Ollivier-type discretization of Ricci curvature for $(F_{\text{NACE}}, A_{\text{NACE}})$

From Figs. 2 and 3, it can be directly seen that the $\mathbb{F}(F_{\text{NACE}}(e))$ and $\mathbb{O}_F(F_{\text{NACE}}(e))$ curvatures have similar distributions which are the variations of the mixture distribution of uniform and normal distributions. However, such similarity is not sufficient to demonstrate the effectiveness and usefulness of the presented method. Therefore, we compare the pairwise Wasserstein-1 distances between the empirical distributions of the $\mathbb{F}(F_{\text{NACE}}(e))$ and $\mathbb{O}_F(F_{\text{NACE}}(e))$ values on $(F_{\text{NACE}}, A_{\text{NACE}})$. We shall note that, this Wasserstein-1 distance is defined on empirical distribution that is different than the one we present in Equation 10. The resulting values are presented in Fig. 4.

3.2 Stock market crisis

Different restrictions on the parameter map of a geometric soft set let us obtain the soft analogues of some crisp computational complexes. As the second example, we consider the daily closure price data of the US stock markets NASDAQ and S&P 500 from mid of 2006 to end of 2012 due to their large size and importance among world capital markets. The stocks with missing data are removed and 77 stocks are selected for NASDAQ and 425 stocks are selected for S&P 500. The companies operating in each stock market are listed in Tables 3 and 4 in Appendix II.

In the first example, we consider one big soft set with different element sizes. In this second example, we first form a 4-regular soft set for each stock markets, then study the changes of the values of discrete Ricci curvatures defined on them. The time scale of our analysis is obtained by subdividing the whole time span into 12 equal length of 151 sub-time interval as they cover pre- and post-periods of the global economic crisis of 2008.

For the preprocessing of the closure price data, we first compute the logarithmic returns of the daily closure prices as $Cl_i = \log(r_i(t+1)) - \log(r_i(t))$, where $r_i(t)$ is the closure price of the stock i at t. Then, we compute the Pearson correlation coefficient between the stock i and j with

$$\rho_{ij} = \frac{< Cl_i Cl_j > - < Cl_i >< Cl_j >}{\sqrt{< Cl_i^2 - < Cl_i >^2 >< Cl_j^2 - < Cl_j >^2 >}}. \qquad (21)$$

In order to determine the which stocks are correlated most, we use the correlation distance between each stocks by

$$d_C(i, j) = \sqrt{2(1 - \rho_{ij})}. \qquad (22)$$

The correlation distance matrices for NASDAQ and S&P500 stock markets are presented in Figs. 6 and 7 in Appendix II. Our analysis uses a moving time window in order to capture structural changes on parameters. But in aforementioned figures, we present the data matrices for complete time scale. In those figures, it can be seen that companies in stock markets tend to form clusters. Hence, we determine the parameter map assigning heuristic to dependent on the correlation distance-based clustering. In the parameter map assigning heuristic, we first determine the quadruple of stocks which has the lowest total d_C value and assign the first parameter with weight of the mean value of d_C among them. Then, we add other stock to triple in the first parameter by determining the score is the minimum of the total d_C value in order to obtain the second parameter. At the end of the process, we obtain a weighted soft set at each time step t whose parameters are assigned to 4-elements. Then, the Forman-Ricci curvatures can be computed by using Equation 8 and Ollivier-Ricci curvatures can be computed by using the Equation 20. For the computation of Ollivier-Ricci curvatures, we use the total d_C values as the weight of a parameter.

The computational results of the mean values of $\mathbb{F}$ and $\mathbb{O}_F$ on soft sets emerging from NASDAQ and S&P500 data sets denoted by (F_{NASDAQ}, E_{NASDAQ}) and $(F_{S\&P500}, E_{S\&P500})$, respectively, are presented in Fig. 5.

In this section, we form these representations of NASDAQ and S&P500 stock markets, which are leading markets in global scale, and analyzed the discrete Ricci curvatures of parameters through out an economic crisis. These stock market examples are weighted soft sets by their definition.

4 Results and discussions

Real-world systems are often modeled with agents in nonlinear relationships. The strengths and transients of these relationships require new methods to be used in the modeling process. Especially in modeling social systems, uncertainties between relationships and fault tolerance should be taken into account in the analysis of the system. In recent years, the use of soft sets in the analysis of systems has often come across as a soft computing technique. In general, we can consider soft sets as the set system formed by parameterizing an initial universe. The biggest criticism in soft set calculations is that these set systems can be examined with a set-valued mapping. However, since the set system formed by soft sets includes the relationship between the parameters, a topological and geometrical analysis on the parameters reveals the effectiveness of this method. In this study, we extend the Forman and Ollivier- property relaxed, then we present statistical analyses.

Laplacian operator is defined on geometric soft sets to extend the Forman-Ricci curvature to soft sets. This operator is defined on the co-chain groups of soft p-faces parameterized by the incidence relationship. The Forman-Ricci curvature is then extended to soft sets by the combinatorial decomposition of the described Laplacian operator. Weights of parameters play an important role in parameter modeling of real-world systems. For this reason, an extension is made that includes the weights of the parameters. Forman-Ricci curvature in soft clusters is then defined according to the weighted geodesic and volumetric growth of the parameters. Extending of the Ollivier-Ricci curvature to soft sets is considered as the transport problem of topological balls formed by parameter sets. To define this type of curvature, two probability measures defined in soft sets are first defined. The first presented probability measure is defined on soft p-faces in the geometrization of soft sets. With this probability measure, the Ollivier-Ricci curvature is extended by the distance between the probability distributions of parameters between. However, since this

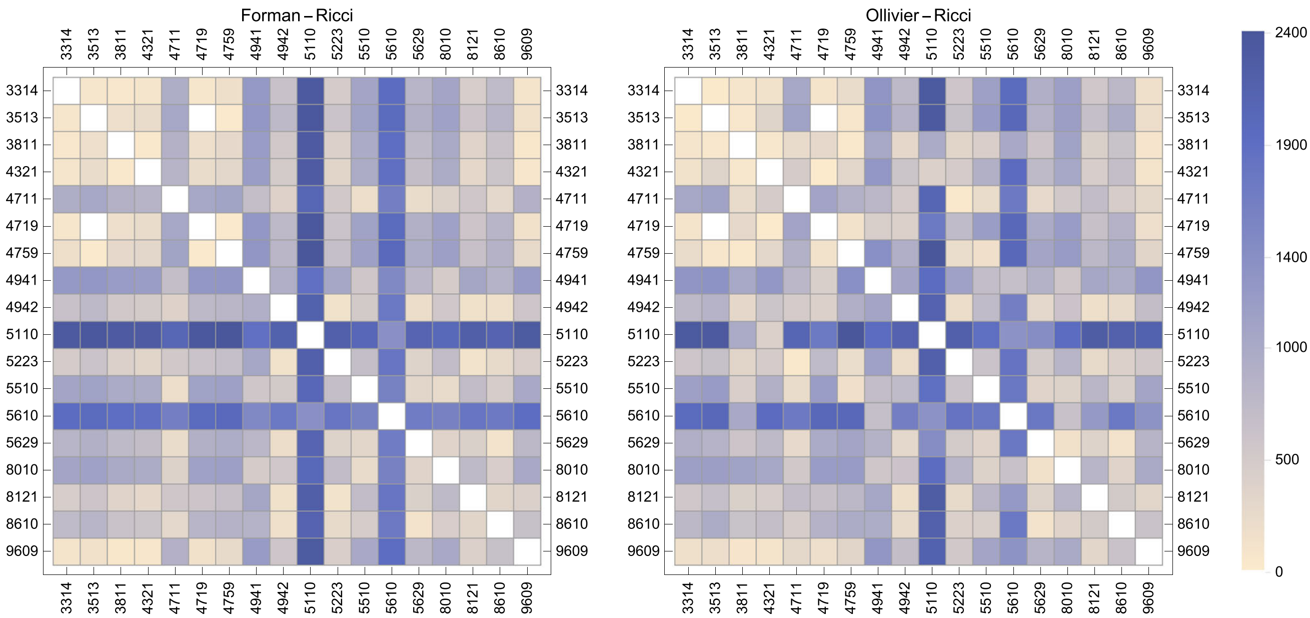

Fig. 4 Wasserstein-1 distances of the empirical distributions of the discrete Ricci curvatures

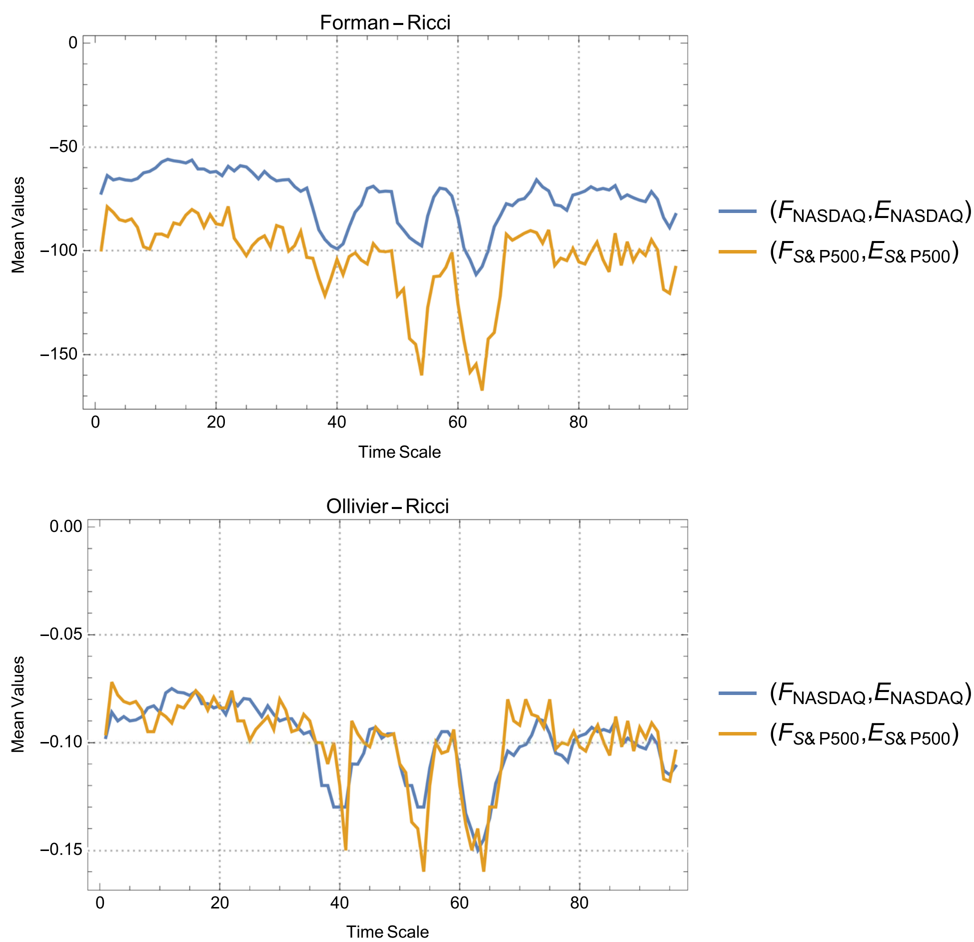

Fig. 5 Mean $\mathbb{F}$ and $\mathbb{O}_F$ values on (F_{NASDAQ}, E_{NASDAQ}) and $(F_{S\&P500}, E_{S\&P500})$ through out the economic crisis of 2008

approach is defined according to two parameters, it does not contribute much to the realization of the hypothesis about the effectiveness of the method. The second measurement of probability that we present is defined on the topological structure established on the parameters. With this probability measure, an Ollivier-Ricci curvature extension is made, which turns to be multi-marginal optimal transport problem. Since the solution of the multi-marginal optimal transport problem is a linear program, the Ollivier-Ricci curvatures are calculated by the barycenters of duals of soft sets.

In this study, two real-world examples are discussed to demonstrate the effectiveness of the novel method we present and to guide the modeling of multi-agent systems with soft sets. The first of these examples is soft set models of occupational accidents occurred in Turkey during 2013–2014. In the occupational accident example, soft sets according to each sector are created by grading the parameters in the occupational accident data. Each of the soft sets created for a total of 18 sectors are unweighted soft sets that is we do not involve any strength to the graded parameters. These soft clusters can be weighted with quantitative data such as treatment cost of work accident. In the example, each sector is expressed with the NACE code. Forman and Ollivier-type Ricci curvatures presented on the soft sets of each sector are calculated. Discrete Ricci curvatures histograms are given in Figs. 2 and 3. As can be seen from these figures, the discrete Ricci curvature of both types has mostly received negative values. It is also observed from the figures that these negative values are often quite small. Therefore, it can be said that topological balls are far from the centers in these soft set models presented geometrically. Moreover, parameters with a larger negative Ollivier-Ricci curvature value show more effective parameters in the soft set model. It will be effective to handle such parameters in a possible decision-making process, especially by policy makers. Similarities of distributions of Forman and Ollivier-type Ricci curvatures can be seen from histograms. However, such a similarity is not sufficient for statistical analysis alone. For this reason, in our study, the distributions of Forman and Ollivier-type Ricci curvatures in soft set models of sectors where occupational accidents occurred are compared. This comparison is made according to the Wasserstein-1 distance of empirical distributions. As a result, the sectors where the most similar work accidents occurred are found as "Distribution of Electricity" and "Other retail sale in non-specialized stores." This result is consistent with the results obtained in studies (Balci and Tuna 2018; Tuna and Kurt 2017).

In the soft set model of the second multi-agent system discussed in this study, the elements of the initial universe are taken as companies with intensive relationships traded in stock markets. The relationships of the companies traded in the stock markets are determined by the correlations of the time series expressed in the logarithmic returns of the daily closing prices. In our study, $NASDAQ$ and $S\&P500$ data, which are the two leading stock markets of the USA due to their dominant characters in the world economy, are discussed. Structures determined by correlation distances in stocks tend to cluster. For this reason, while determining the parameters in the soft set modeling of the stock markets, the clustering tendency of companies with the closest correlation distances is selected. In this type of approach, the cardinality of the subsets determines the size of the soft set to be formed. High cardinal subsets will produce statistically ineffective results as they will form very high-dimensional soft sets. In our study, to keep dimensions of soft sets low, a maximum of 4-element subsets which have geometric realization in 3-dimensional Euclidean space are selected. In stress situations such as the economic crisis, because there are structural changes in financial multi-agent systems, Forman and Ollivier-type Ricci curves are calculated and analyzed in soft set models for the analysis and even control of this change. Our data set is divided into 96 moving windows to center the global economic crisis that occurred in 2008. Forman and Ollivier-type Ricci curvatures of soft clusters formed in each window are calculated. For both types of discrete curvature, all values are negative. It can also be seen in Fig. 5 that there is a strong correlation between discrete Ricci curvatures for each sliding window. Considering the 2008 global economic crisis, the averages of both types of discrete Ricci curves increase negatively. Hence, we can say that discrete Ricci curvature values are important indicators for these kind of soft set models of stock markets.

This novel method, which can be used especially in the modeling of multi-agent systems, has certain limitations. In the definition of both curvatures, duals of soft sets are considered. Calculation of Forman or Ollivier-type Ricci curvatures will yield statistically insufficient results, especially in soft set models that are sparse in terms of parameter cardinality. In addition, the high computational complexity of the Ollivier-Ricci curvature presents a serious disadvantage over the Forman-Ricci curvature. Although we have presented a barycentric computational method for the Ollivier-Ricci curvature in our study, it takes a serious workload to calculate these barycenters. To prevent this type of problem, ISOMAP type size dimensional reduction algorithms of soft sets can be produced.

5 Conclusions

The soft set theory is a mathematical tool dealing with the uncertainty of real-world problems which usually contain uncertain data, and depends on the adequacy of the parametrization. Hence, if the elements of the universe set have a geometric realization, the geometric analysis of such soft sets regarding to parameter mapping becomes an important subject. In this study, we extend the idea of discrete Ricci curvatures defined on cellular complexes to soft sets. Moreover, in order to analyze real-world problems, we analogously define Forman and Ollivier-type Ricci curvatures on general soft sets by considering the volumes of the parameters as their cardinality and the distance between new probability measures defined on soft sets.

On the basis of the contributions presented in this paper, several promising lines are still open for further research on different types of multi-agent system environments such as biological systems, computer communication systems, or further financial systems. Besides, more geometric analysis on the flows of the discrete Ricci curvatures on soft sets is still an open problem. Moreover, we point to the algebraic concept called soft free Abelian groups dimensional reduction and Euclidean embedding algorithms for researchers working on soft set theory and its use in soft computing. It is also well known that the lower Ricci curvature bounds estimate the tendency of geodesics to converge. Hence, for further studies, determining the Forman or Ollivier-type Ricci curvature lower boundaries on soft sets will be helpful to give more characterization to different types of systems.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Appendices

Appendix 1: NACE codes

See the Table 2.

Table 2 NACE codes and labels

NACE code	Labels
3314	Repair of electrical equipment
3513	Distribution of electricity
3811	Collection of non-hazardous waste
4321	Electrical installation
4711	Retail sale in non-specialized stores
4719	Other retail sale in non-specialized stores
4759	Retail sale in specialized stores
4941	Freight transport by road
4942	Removal services
5110	Passenger air transport
5223	Service activities incidental to air transportation
5510	Hotels and similar accommodation
5610	Restaurants and mobile food service activities
5629	Other food service activities
8010	Private security activities
8121	General cleaning of buildings
8610	Hospital activities
9609	Other personal service activities n.e.c.

Appendix 2: Data for NASDAQ and *S&P*500

See the Tables 3, 4 and Figs. 6, 7.

Table 3 Tickers of the companies operating in NASDAQ

ATVI	ADBE	AKAM	ALXN	AMZN	AMGN	ADI
AAPL	AMAT	ADSK	ADP	BIIB	BMRN	CA
CELG	CERN	CHKP	CTAS	CSCO	CTXS	CTSH
CMCSA	COST	CSX	CTRP	XRAY	DISH	DLTR
EBAY	EA	ESRX	FAST	FISV	GILD	HAS
HSIC	HOLX	ILMN	INCY	INTC	INTU	ISRG
KLAC	LRCX	MAR	MAT	MXIM	MCHP	MU
MSFT	MDLZ	MNST	MYL	NTES	NVDA	ORLY
PCAR	PAYX	QCOM	REGN	ROST	SBAC	STX
SHPG	SWKS	SBUX	SYMC	TXN	PCLN	TSCO
FOXA	FOX	VRTX	VOD	WBA	WDC	XLNX

Table 4 Tickers of the companies operating in $S\&PP500$

AAP	AAPL	ABC	ABT	ACN	ADBE	ADI	ADM	ADP	ADS
ADSK	AEE	AEP	AES	AET	AFL	AGN	AIG	AIV	AJG
AKAM	ALB	ALK	ALL	ALXN	AMAT	AME	AMG	AMGN	AMT
AMZN	AN	ANTM	AON	APA	APC	APD	APH	ARNC	ATVI
AVB	AVY	AXP	AYI	AZO	BA	BAC	BAX	BBBY	BBT
BBY	BCR	BDX	BEN	BHI	BIIB	BK	BLK	BLL	BMY
BSX	BWA	BXP	C	CA	CAG	CAH	CAT	CB	CCI
CCL	CELG	CERN	CHD	CHK	CHRW	CI	CINF	CL	CLX
CMA	CMCSA	CME	CMI	CMS	CNC	CNP	COF	COG	COH
COL	COO	COP	COST	CPB	CSCO	CSX	CTAS	CTL	CTSH
CTXS	CVS	CVX	D	DD	DE	DGX	DHI	DHR	DIS
DLTR	DNB	DOV	DOW	DRI	DTE	DUK	DVA	DVN	EA
EBAY	ECL	ED	EFX	EIX	EL	EMN	EMR	EOG	EQIX
EQR	EQT	ES	ESRX	ESS	ETFC	ETN	ETR	EW	EXC
EXPD	F	FAST	FCX	FDX	FE	FFIV	FIS	FISV	FITB
FL	FLIR	FLR	FLS	FMC	FOX	FOXA	FRT	FTI	FTR
GD	GE	GGP	GILD	GIS	GLW	GPC	GPN	GPS	GRMN
GS	GT	GWW	HAL	HAS	HBAN	HCN	HCP	HD	HES
HIG	HOG	HOLX	HON	HP	HPQ	HRB	HRL	HRS	HSIC
HST	HSY	HUM	IBM	IDXX	IFF	ILMN	INCY	INTC	INTU
IP	IPG	IR	IRM	ISRG	ITW	IVZ	JBHT	JCI	JEC
JNJ	JNPR	JPM	JWN	K	KEY	KIM	KLAC	KMB	KMX
KO	KR	KSS	KSU	L	LB	LEG	LEN	LH	LKQ
LLL	LLY	LMT	LNC	LNT	LOW	LRCX	LUK	LUV	LVLT
M	MAA	MAC	MAR	MAS	MAT	MCD	MCHP	MCK	MCO
MDLZ	MDT	MET	MHK	MKC	MLM	MMC	MMM	MNST	MO
MON	MOS	MRK	MRO	MS	MSFT	MSI	MTB	MTD	MU
MUR	MYL	NBL	NDAQ	NEE	NEM	NFLX	NFX	NI	NKE
NOC	NOV	NRG	NSC	NTAP	NTRS	NUE	NVDA	NWL	O
OKE	OMC	ORCL	ORLY	OXY	PAYX	PBCT	PCAR	PCG	PCLN
PDCO	PEG	PEP	PFE	PFG	PG	PGR	PH	PHM	PKI
PLD	PNC	PNR	PNW	PPG	PPL	PRGO	PRU	PSA	PVH
PWR	PX	PXD	QCOM	R	RAI	RCL	REG	REGN	RF
RHI	RHT	RIG	RL	ROK	ROP	ROST	RRC	RSG	RTN
SBUX	SCG	SCHW	SEE	SHW	SIG	SJM	SLB	SLG	SNA
SO	SPG	SPGI	SPLS	SRCL	SRE	STI	STT	STX	STZ
SWK	SWKS	SWN	SYK	SYMC	SYY	T	TAP	TGNA	TGT
TIF	TJX	TMK	TMO	TROW	TRV	TSCO	TSN	TSO	TSS
TWX	TXN	TXT	UDR	UHS	UNH	UNM	UNP	UPS	URBN
URI	USB	UTX	VAR	VFC	VLO	VMC	VNO	VRSN	VRTX
VTR	VZ	WAT	WBA	WDC	WEC	WFC	WFM	WHR	WLTW
WM	WMB	WMT	WY	WYNN	XEC	XEL	XL	XLNX	XOM
XRAY	XRX	YUM	ZBH	ZION					

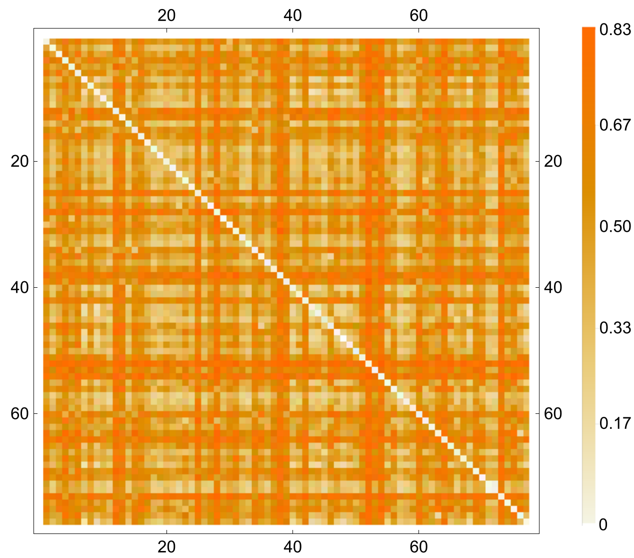

Fig. 6 Correlation distance matrix of NASDAQ

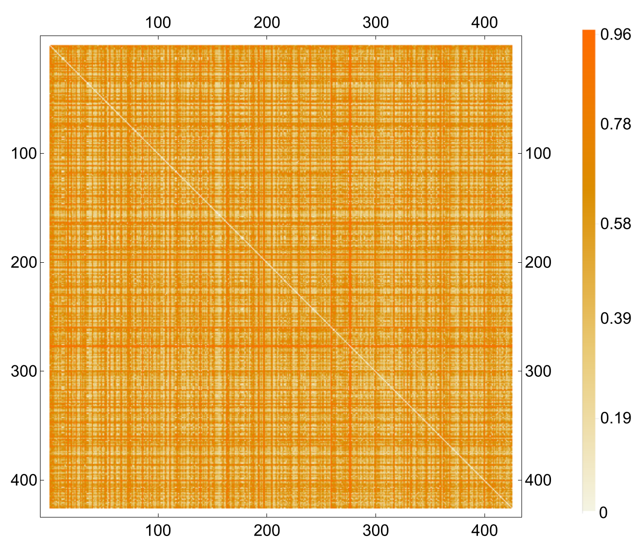

Fig. 7 Correlation distance matrix of $S\&P\,P500$

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