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A Taylor method for numerical solution of generalized pantograph equations with linear functional argument

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Abstract

This paper is concerned with a generalization of a functional differential equation known as the pantograph equation which contains a linear functional argument. In this paper, we introduce a numerical method based on the Taylor polynomials for the approximate solution of the pantograph equation with retarded case or advanced case. The method is illustrated by studying the initial value problems. The results obtained are compared by the known results.

MSC: 34K06; 34K28

Keywords: Pantograph equations; Functional equations; Taylor method

1. Introduction

Taylor methods have been given to solve linear differential, integral and integro-differential equations with approximate and exact solutions [15,18,21,24]. In recent years, many papers have been devoted to problem of approximate solution of difference, differential-difference and integro-difference equations [10–12,22]. Our purpose in this study is to develop and to apply mentioned methods to the generalized pantograph equation

$$y^{(m)}(t) = \sum_{j=0}^{J} \sum_{k=0}^{m-1} P_{jk}(t) y^{(k)}(\alpha_j t + \beta_j) + f(t) \tag{1}$$

which is a generalization of the pantograph equations given in [4,8,16,17] with the initial conditions

$$\sum_{k=0}^{m-1} c_{ik} y^{(k)}(0) = \lambda_i, \quad i = 0, 1, \ldots, m-1 \tag{2}$$

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and to find the solution in terms of the Taylor polynomial form, in the origin,

$$y(t)=\sum_{n=0}^{N} y_n t^n, \quad y_n=\frac{y^{(n)}(0)}{n!}. \tag{3}$$

Here $P_{jk}(t)$ and $f(t)$ are analytical functions; c_{ik}, λ_i, α_j and β_j are real or complex constants; the coefficients y_n, $n=0,1,\ldots,N$ are Taylor coefficients to be determined.

In recent years, there has been a growing interest in the numerical treatment of pantograph equations of the retarded and advanced type. A special feature of this type is the existence of compactly supported solutions [4]. This phenomenon was studied in [3] and has direct applications to approximation theory and to wavelets [5].

Pantograph equations are characterized by the presence of a linear functional argument and play an important role in explaining many different phenomena. In particular they turn out to be fundamental when ODEs-based model fail. These equations arise in industrial applications [9,19] and in studies based on biology, economy, control and electrodynamics [1,2].

2. Fundamental relations

Let us convert expressions defined in (1)–(3) to the matrix forms. Let us first assume that the functions $y(t)$ and its derivative $y^{(k)}(t)$ can be expanded to Taylor series about $t=0$ in the form

$$y^{(k)}(t)=\sum_{n=0}^{\infty} y_n^{(k)} t^n, \tag{4}$$

where for $k=0$, $y^{(0)}(t)=y(t)$ and $y_n^{(0)}=y_n$.

Now, let us differentiate expression (4) with respect to t and then put $n\to n+1$

$$y^{(k+1)}(t)=\sum_{n=1}^{\infty} n y_n^{(k)} t^{n-1}=\sum_{n=0}^{\infty}(n+1)y_{n+1}^{(k)} t^n. \tag{5}$$

It is clear, from (4), that

$$y^{(k+1)}(t)=\sum_{n=0}^{\infty} y_n^{(k+1)} t^n. \tag{6}$$

Using relations (5) and (6), we have the recurrence relation between the Taylor coefficients of $y^{(k)}(t)$ and $y^{(k+1)}(t)$

$$y_n^{(k+1)}=(n+1)y_{n+1}^{(k)}, \quad n,k=0,1,2,\ldots\ . \tag{7}$$

If we take $n=0,1,\ldots,N$ and assume $y_n^{(k)}=0$ for $n>N$, then we can transform system (7) into the matrix form

$$\mathbf{Y}^{(k+1)}=\mathbf{M}\mathbf{Y}^{(k)}, \quad k=0,1,2,\ldots, \tag{8}$$

where

$$\mathbf{Y}^{(k)}=\begin{bmatrix} y_0^{(k)} \\ y_1^{(k)} \\ \vdots \\ y_N^{(k)} \end{bmatrix}, \quad \mathbf{M}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

For $k=0,1,2,\ldots$, it follows from relation (8) that

$$\mathbf{Y}^{(k)}=\mathbf{M}^k\mathbf{Y}, \tag{9}$$

where clearly

$$\mathbf{Y}^{(0)} = \mathbf{Y} = [y_0 \ \ y_1 \ \ \cdots \ \ y_N]^{\mathrm{T}}.$$

On the other hand, solution expressed by (3) and its derivatives can be written in the matrix forms

$$y(t) = \mathbf{TY} \quad \text{and} \quad y^{(k)}(t) = \mathbf{TY}^{(k)}$$

or using relation (9)

$$y^{(k)}(t) = \mathbf{TM}^k\mathbf{Y}, \tag{10}$$

where

$$\mathbf{T} = [1 \ \ t \ \ t^2 \ \ \ldots \ \ t^N].$$

To obtain the matrix form of the part

$$\sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}(t)y^{(k)}(\alpha_j t + \beta_j) \tag{11}$$

which is defined in Eq. (1), we first write the function $P_{jk}(t)$ in the form

$$P_{jk}(t) = \sum_{i=0}^{N} p_{jk}^{(i)} t^i, \quad p_{jk}^{(i)} = \frac{P_{jk}^{(i)}(0)}{i!}$$

and then, substitute into (11). It is seen from relation (4) and binomial expansion that

$$y^{(k)}(\alpha_j t + \beta_j) = \sum_{n=0}^{N} y_n^{(k)}(\alpha_j t + \beta_j)^n = \sum_{n=0}^{N}\sum_{r=0}^{n}\binom{n}{r}\alpha_j^{n-r}\beta_j^r t^{n-r} y_n^{(k)}.$$

Thus, the term $t^i y^{(k)}(\alpha_j t + \beta_j)$ is obtained and its matrix representations becomes

$$t^i y^{(k)}(\alpha_j t + \beta_j) = \sum_{n=0}^{N}\sum_{r=0}^{n}\binom{n}{r}\alpha_j^{n-r}\beta_j^r t^{n-r+i} y_n^{(k)} = \mathbf{TI}_i\mathbf{A}_j\mathbf{Y}^{(k)}$$

or from (9)

$$t^i y^{(k)}(\alpha_j t + \beta_j) = \mathbf{TI}_i\mathbf{A}_j\mathbf{M}^k\mathbf{Y}, \quad i = 0, 1, \ldots, N, \tag{12}$$

where

$$I_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad I_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad \ldots, \quad I_N = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix};$$

for $\beta_j \neq 0$,

$$A_j = \begin{bmatrix} \binom{0}{0}(\alpha_j)^0(\beta_j)^0 & \binom{1}{1}(\alpha_j)^0(\beta_j)^1 & \cdots & \binom{N}{N}(\alpha_j)^0(\beta_j)^N \\ 0 & \binom{1}{0}(\alpha_j)^1(\beta_j)^0 & \cdots & \binom{N}{N-1}(\alpha_j)^1(\beta_j)^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N}{0}(\alpha_j)^N(\beta_j)^0 \end{bmatrix};$$

and for $\beta_j = 0$,

$$\mathbf{A}_j = \begin{bmatrix} (\alpha_j)^0 & 0 & \cdots & 0 \\ 0 & (\alpha_j)^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\alpha_j)^N \end{bmatrix}.$$

Utilizing expression (12), we obtain the matrix form of the part (11) as

$$\sum_{j=0}^{J} \sum_{k=0}^{m-1} \sum_{i=0}^{N} p_{jk}^{(i)} \mathbf{T}\mathbf{I}_i \mathbf{A}_j \mathbf{M}^k \mathbf{Y}. \tag{13}$$

We now assume that the function $f(t)$ can be expanded as

$$f(t) = \sum_{n=0}^{N} f_n t^n, \quad f_n = \frac{f^{(n)}(0)}{n!}$$

or written in the matrix form

$$f(t) = \mathbf{TF}, \tag{14}$$

where $\mathbf{F} = [f_0 \;\; f_1 \;\; \ldots \;\; f_N]^{\mathrm{T}}$.

Next, by means of relation (10), we can obtain the corresponding matrix form for the initial conditions (2) as

$$\sum_{k=0}^{m-1} c_{ik} \mathbf{T}(0) \mathbf{M}^k \mathbf{Y} = \lambda_i, \quad i = 0, 1, \ldots, m-1, \tag{15}$$

where

$$\mathbf{T}(0) = [1 \;\; 0 \;\; 0 \;\; \cdots \;\; 0].$$

3. Method of solution

We are now ready to construct the fundamental matrix equation corresponding to Eq. (1). For this purpose, substituting matrix relations (10), (13) and (14) into Eq. (1) and then simplifying, we obtain the fundamental matrix equation

$$\left\{ \mathbf{M}^m - \sum_{j=0}^{J} \sum_{k=0}^{m-1} \sum_{i=0}^{N} p_{jk}^{(i)} \mathbf{I}_i \mathbf{A}_j \mathbf{M}^k \right\} \mathbf{Y} = \mathbf{F} \tag{16}$$

which corresponds to a system of $(N+1)$ algebraic equations for the $(N+1)$ unknown coefficients $y_0, y_1, \ldots, y_N$. Briefly, we can write Eq. (16) in the form

$$\mathbf{WY} = \mathbf{F} \quad \text{or} \quad [\mathbf{W}; \mathbf{F}],$$

where

$$\mathbf{W} = [w_{nh}], \quad n, h = 0, 1, \ldots, N.$$

Also, the matrix form (15) for conditions (2) can be written as

$$\mathbf{U}_i \mathbf{Y} = \lambda_i \quad \text{or} \quad [\mathbf{U}_i; \lambda_i], \quad i = 0, 1, \ldots, m-1,$$

where

$$\mathbf{U}_i = \sum_{k=0}^{m-1} c_{ik} \mathbf{T}(0) \mathbf{M}^k = [u_{i0} \;\; u_{i1} \;\; \cdots \;\; u_{iN}].$$

To obtain solution of Eq. (1) under conditions (2), by replacing the m rows matrices $[\mathbf{U}_i; \lambda_i]$ by the last m rows of the matrix $[\mathbf{W}; \mathbf{F}]$, we have the augmented matrix

$$[\tilde{\mathbf{W}}; \tilde{\mathbf{F}}] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & f_0 \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & f_1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} & ; & f_{N-m} \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & ; & \lambda_{m-1} \end{bmatrix}.$$

If det $\tilde{\mathbf{W}} \neq 0$, then we can write

$$\mathbf{Y} = (\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{F}}.$$

Thus the coefficients y_n, $n = 0, 1, \ldots, N$ are uniquely determined by this equation.

We can easily check the accuracy of the solutions as follows. Since the obtained polynomial solution is an approximate solution of Eq. (1), it must be satisfied approximately; that is, for $t = t_r$, $r = 0, 1, \ldots$

$$E(t_r) = \left| y^{(m)}(t_r) - \sum_{j=0}^{J} \sum_{k=0}^{m-1} P_{jk}(t_r) y^{(k)}(\alpha_j t_r + \beta_j) - f(t_r) \right| \cong 0 \tag{17}$$

or $E(t_r) \leqslant 10^{-k_r}$ (k_r any positive integer) is prescribed, then the truncation limit N is increased until the difference $E(t_r)$ at each of the points t_r becomes smaller than the prescribed 10^{-k}.

4. Illustrative examples

In this section, several numerical examples are given to illustrate the properties of the method and all of them were performed on the computer using a program written in Mathcad 2001 Professional. The absolute errors in Tables 2–4 are the values of $|y(x) - y_N(x)|$ at selected points.

Example 1. Consider the linear delay differential equation of first order

$$y'(t) = -y(0.8t) - y(t), \quad y(0) = 1. \tag{18}$$

From Eq. (16), the fundamental matrix equation of the problem is

$$(\mathbf{M} + \mathbf{I}_0 + \mathbf{A}_1)\mathbf{Y} = \mathbf{F},$$

where $\mathbf{I}_0$ is unit matrix, $\mathbf{M}$ and $\mathbf{A}_1$ for $\alpha_1 = 0.8$, $\beta_1 = 0$ are defined in relations (8) and (12), respectively.

Table 1 shows solutions of Eq. (18) with $N = 8$, 11 and 19 by presented method. The previous results of Rao and Palanisamy by Walsh series approach [20], Hwang by delayed unit step function (DUSF) series approach [13], and Hwang and Shih by Laguerre series approach [14] are also given in Table 1 for comparison. The Taylor method seems more rapidly convergent than Laguerre series, and with errors more under control than for the Walsh or DUSF series.

Example 2. Consider the following problem:

$$y'(t) = \frac{1}{2} e^{t/2} y\left(\frac{t}{2}\right) + \frac{1}{2} y(t), \quad y(0) = 1, \quad 0 \leqslant t \leqslant 1 \tag{19}$$

which has the exact solution $y(t) = e^t$.

Table 1
Comparison of the solutions of Eq. (18)

t	Walsh series method	DUSF series method	Laguerre series method		Taylor series method			
		$m=100, h=0.01$	$n=20$	$n=30$	$N=8$	$N=11$	$N=19$, $y(t)$	$N=19$, $E(t)$
0	1.000000	1.000000	0.999971	1.000000	1.000000	1.000000	1.000000000000000	8.44 E − 15
0.2	0.665621	0.664677	0.664703	0.664691	0.664691	0.664691	0.664691000828909	1.38 E − 14
0.4	0.432426	0.433540	0.433555	0.433561	0.433561	0.433561	0.433560778776339	3.22 E − 14
0.6	0.275140	0.276460	0.276471	0.276482	0.276483	0.276482	0.276482330222267	1.25 E − 14
0.8	0.170320	0.171464	0.171482	0.171484	0.171494	0.171484	0.171484111976062	7.38 E − 15
1	0.100856	0.102652	0.102679	0.102670	0.102744	0.102670	0.102670126574418	1.55 E − 14

Table 2
Comparison of the absolute errors for Eq. (19)

t	Spline method, $h=0.001$			Adomian method with 13 terms [8]	Present method			
	$m=2$ [6]	$m=3$ [23]	$m=4$ [7]		$N=8$	$N=12$	$N=15$	$N=16$
0.2	0.198 E − 7	1.37 E − 11	3.10 E − 15	0.00	1.440 E − 12	2.220 E − 16	2.220 E − 16	2.22 E − 16
0.4	0.473 E − 7	3.27 E − 11	7.54 E − 15	2.22 E − 16	7.524 E − 10	1.332 E − 15	2.220 E − 16	2.22 E − 16
0.6	0.847 E − 7	5.86 E − 11	1.39 E − 14	2.22 E − 16	2.953 E − 8	2.189 E − 13	2.220 E − 16	2.22 E − 16
0.8	0.135 E − 6	9.54 E − 11	2.13 E − 14	1.33 E − 15	4.018 E − 7	9.361 E − 12	1.332 E − 15	0.00
1	0.201 E − 6	1.43 E − 10	3.19 E − 14	4.88 E − 15	3.059 E − 6	1.729 E − 10	5.018 E − 14	2.22 E − 15

When the presented method is applying to Eq. (19), the fundamental matrix equation becomes

$$\left(\mathbf{M}-\frac{1}{2}\mathbf{I}_0-\sum_{i=0}^{N}p^{(i)}\mathbf{I}_i\mathbf{A}_1\right)\mathbf{Y}=\mathbf{F},$$

where $p^{(i)}$ is the Taylor coefficients of $1/2\mathrm{e}^{t/2}$, $\mathbf{A}_1$ for $\alpha_1=0.5$, $\beta_1=0$ and $\mathbf{I}_i$ are defined in relation (12). Hence, the computed results are compared with other methods [6–8,23] in Table 2. The Taylor method has better results than the spline methods for different N. However, the absolute errors of Adomian and the Taylor methods seem like each other.

Example 3. Consider the pantograph equation of first order

$$y'(t)=-y(t)+\frac{q}{2}y(qt)-\frac{q}{2}\mathrm{e}^{-qt},\quad y(0)=1, \tag{20}$$

where $y(t)=\mathrm{e}^{-t}$. Table 3 compares the results of the present method and the collocation method [17] for this problem. Note that $q=1$ is not a pantograph equation, is a linear differential equation. In any case, the Taylor method has far better results than collocation method.

Example 4. Consider the pantograph equation with variable coefficients

$$y'(t)=-y(t)+\mu_1(t)y(t/2)+\mu_2(t)y(t/4),\quad y(0)=1.$$

Here $\mu_1(t)=-\mathrm{e}^{-0.5t}\sin(0.5t)$, $\mu_2(t)=-2\mathrm{e}^{-0.75t}\cos(0.5t)\sin(0.25t)$. It can be seen that the exact solution of this problem is $y(t)=\mathrm{e}^{-t}\cos(t)$ [16]. Using the method with $N=7$, we obtain the approximate solution

$$y(t)=1-t+\frac{1}{3}t^3-\frac{1}{6}t^4+\frac{3333333333333}{99999999999991}t^5-\frac{6105006105}{3846153846149}t^7.$$

Note that the coefficients $y_0, y_1, y_2, y_3, y_4, y_6$ are the same as the exact solution and others are the same till the 15 decimal place.

Table 3
Comparison of the absolute errors for Eq. (20)

t	$q=1$			$q=0.2$		
	Collocation M.	Present method		Collocation M.	Present method	
	$m=2$	$N=6$	$N=13$	$m=2$	$N=6$	$N=13$
2^{-1}	5.005 E − 06	1.458 E − 06	7.772 E − 16	2.719 E − 05	1.458 E − 06	7.772 E − 16
2^{-2}	1.877 E − 07	1.174 E − 08	1.110 E − 16	1.080 E − 06	1.174 E − 08	1.110 E − 16
2^{-3}	6.434 E − 09	9.315 E − 11	2.220 E − 16	3.817 E − 08	9.315 E − 11	2.220 E − 16
2^{-4}	2.106 E − 10	7.334 E − 13	0.000	1.269 E − 09	7.334 E − 13	0.000
2^{-5}	6.700 E − 12	5.662 E − 15	1.110 E − 16	4.090 E − 11	5.662 E − 15	1.110 E − 16
2^{-6}	2.100 E − 13	0.000	0.000	1.200 E − 12	0.000	0.000

Example 5 (*Evans and Raslan, [8]*). Consider the pantograph equation of second order

$$y''(t) = \frac{3}{4}y(t) + y\left(\frac{t}{2}\right) - t^2 + 2, \quad y(0)=0, \quad y'(0)=0, \quad 0 \leqslant t \leqslant 1.$$

The fundamental matrix equation of this problem is

$$(\mathbf{M}^2 - \tfrac{3}{4}\mathbf{I}_0 - \mathbf{A}_1)\mathbf{Y} = \mathbf{F}.$$

Here $\mathbf{I}_0$ is unit matrix and for $N=4$ others

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1/8 & 0 \\ 0 & 0 & 0 & 0 & 1/16 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

After the ordinary operations and following the method in Section 3, the augmented matrix for the problem is gained as

$$[\tilde{\mathbf{W}}; \tilde{\mathbf{F}}] = \begin{bmatrix} -7/4 & 0 & 2 & 0 & 0 & ; & 2 \\ 0 & -5/4 & 0 & 6 & 0 & ; & 0 \\ 0 & 0 & -1 & 0 & 12 & ; & -1 \\ 1 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 1 & 0 & 0 & 0 & ; & 0 \end{bmatrix},$$

where the last two rows indicates the augmented matrix of the conditions $[\mathbf{U}_i; \lambda_i]$. Solving this system, we get $y(t)=t^2$ and this is the exact solution. If we take more terms of the Taylor series, we also obtain the same result.

Example 6. Consider the pantograph equation of third order

$$\begin{aligned} &y'''(t) = -y(t) - y(t-0.3) + \mathrm{e}^{-t+0.3}, \quad 0 \leqslant t \leqslant 1 \\ &y(0)=1, \quad y'(0)=-1, \quad y''(0)=1, \quad y(t)=\mathrm{e}^{-t}, \quad t \leqslant 0. \end{aligned} \tag{21}$$

In Table 4, we make a comparison between Adomian series [8] and present Taylor series methods, and also we give accuracy of the solution in Eq. (17). The Table seems that the Taylor method is not as good as Adomian method for small N, but increasing N, the Taylor method is better than Adomian method.

Example 7. Considering the pantograph equation of third order

$$y'''(t) = ty''(2t) - y'(t) - y\left(\frac{t}{2}\right) + t\cos(2t) + \cos\left(\frac{t}{2}\right), \quad y(0)=1, \quad y'(0)=0, \quad y''(0)=-1$$

Table 4
Numerical analysis of Eq. (21)

t	Absolute errors			Accuracy of the solution	
	Adomian method with six terms	Present method		Present method $E(t)$	
		$N=5$	$N=17$	$N=5$	$N=17$
0	8.52 E − 14	0.00	0.00	4.76 E − 10	1.83 E − 9
0.2	3.83 E − 14	8.54 E − 8	0.00	1.27 E − 3	9.22 E − 10
0.4	1.68 E − 13	5.36 E − 6	2.22 E − 16	9.66 E − 3	8.97 E − 11
0.6	6.00 E − 14	5.95 E − 5	1.11 E − 16	3.12 E − 2	8.90 E − 11
0.8	6.66 E − 15	3.26 E − 4	0.00	7.09 E − 2	4.01 E − 11
1	4.57 E − 14	1.21 E − 3	5.55 E − 17	0.13	1.32 E − 11

as in Section 3, we get the fundamental matrix equation

$$(\mathbf{M}^3 - \mathbf{I}_1\mathbf{A}_2\mathbf{M}^2 + \mathbf{M} + \mathbf{A}_1)\mathbf{Y} = \mathbf{F},$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined in relation (12) for $\alpha_1 = 1/2$, $\beta_1 = 0$ and $\alpha_2 = 2$, $\beta_2 = 0$, respectively.

If we take $N = 6$ and follow the Taylor series method in Section 3, the augmented matrix becomes

$$[\bar{\mathbf{W}};\bar{\mathbf{F}}] = \begin{bmatrix} 1 & 1 & 0 & 6 & 0 & 0 & 0 & ; & 1 \\ 0 & 1/2 & 0 & 0 & 24 & 0 & 0 & ; & 1 \\ 0 & 0 & 1/4 & -9 & 0 & 60 & 0 & ; & -1/8 \\ 0 & 0 & 0 & 1/8 & -44 & 0 & 120 & ; & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & ; & -1 \end{bmatrix}.$$

And also by choosing $N = 8$, we obtain the Taylor coefficient matrix

$$\mathbf{Y} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{24} & 0 & -\frac{17361111111}{12499999999919} & 0 & \frac{1490480}{60096153599} \end{bmatrix}.$$

However the exact solution of this problem is $y(t) = \cos(t)$.

5. Conclusions

A new technique, using the Taylor series, to numerically solve the pantograph equations is presented. It is observed that the method has the best advantage when the known functions in equation can be expanded to Taylor series with converge rapidly. To get the best approximation, we take more terms from the Taylor expansion of functions; that is, the truncation limit N must be chosen large enough.

On the other hand, from Table 1, it may be observed that the solutions found for different N show close agreement for various values of t. In particular, our results in tables are usually better than the other methods. Moreover, approximate solutions of Example 4 and Example 7 show very good agreement with the exact solution, and also in Example 5 we get the exact solution. Besides, tables generally show that closer the zero, better results are obtained. However, more term of the Taylor series is required for accurate calculation for large t.

Another considerable advantage of the method is that Taylor coefficients of the solution are found very easily by using the computer programs.

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