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A Taylor polynomial approach for solving differential-difference equations

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Abstract

The purpose of this study is to give a Taylor polynomial approximation for the solution of *m*th-order linear differential-difference equations with variable coefficients under the mixed conditions about any point. For this purpose, Taylor matrix method is introduced. This method is based on first taking the truncated Taylor expansions of the functions in the differential-difference equations and then substituting their matrix forms into the equation. Hence, the result matrix equation can be solved and the unknown Taylor coefficients can be found approximately. In addition, examples that illustrate the pertinent features of the method are presented, and the results of study are discussed. Also we have discussed the accuracy of the method. We use the symbolic algebra program, Maple, to prove our results.

MSC: 39A10; 41A10; 65Q05

Keywords: Taylor polynomials and series; Taylor polynomial solutions; Taylor matrix method; Differential-difference equations

1. Introduction

In recent years, the studies of differential-difference equations, i.e. equations containing shifts of the unknown function and its derivatives, are developed very rapidly and intensively [1–4,8,11]. Problems involving these equations arise in studies of control theory [4] in determining the expected time for

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the generation of action potentials in nerve cells by random synaptic inputs in the dendrites [3], in the modelling of the activation of a neuron [3], in the works on epidemics and population [8], in the two-body problems in classical electrodynamics in the physical systems whose acceleration depends upon its velocity and its position at earlier instants, and in the formulation of the biological reaction phenomena to X-rays [8]. Also, the differential-difference equations occur frequently as a model in mathematical biology and the physical sciences [11].

A Taylor method for solving Fredholm integral equations has been presented in [5] and then this method has been extended by Sezer to Fredholm integro-differential equations [7] and second-order linear differential [9,10].

In this study, the basic ideas of the above studies are developed and applied to the mth-order linear differential-difference equation (which contains only negative shift in the differentiated term) with variable coefficients [8, pp. 228, 229]

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) + \sum_{r=0}^{R} P_r^*(x)y^{(r)}(x-\tau) = f(x),$$
$$R \leqslant m, \quad \tau > 0, \quad -\tau \leqslant x \leqslant 0, \tag{1}$$

with the mixed conditions

$$\sum_{k=0}^{m-1} [a_{ik}y^{(k)}(a) + b_{ik}y^{(k)}(b) + c_{ik}y^{(k)}(c)] = \mu_i \tag{2}$$

$i = 0(1)(m-1), \ a \leqslant c \leqslant b$ and the solution is expressed in the form

$$y(x) = \sum_{n=0}^{N} \frac{y^{(n)}(c)}{n!}(x-c)^n, \quad a \leqslant c \leqslant b, \ N \geqslant m \tag{3}$$

which is a Taylor polynomial of degree N at $x = c$, where $y^{(n)}(c), \ n = 0(1)N$ are the coefficients to be determined.

Here $P_k(x), \ P_r^*(x)$ and $f(x)$ are functions defined on $a \leqslant x \leqslant b$; the real coefficients $a_{ik}, \ c_{ik}, \ b_{ik}$ and μ_i are appropriate constants.

The rest of this paper is organized as follows. Higher-order linear differential-difference equation with variable coefficients and fundamental relations are presented in Section 2. The new scheme are based on Taylor matrix method. The method of finding approximate solution is described in Section 3. To support our findings, we present result of numerical experiments in Section 4. Section 5 concludes this article with a brief summary.

2. Fundamental relations

Let us consider the linear differential-difference equation with variable coefficients (1) and find the truncated Taylor series expansions of each term in expression (1) at $x=c$ and their matrix representations. We first consider the desired solution $y(x)$ of Eq. (1) defined by a truncated Taylor series (3). Then we can put series (3) in the matrix form

$$[y(x)] = \mathbf{X}\mathbf{M}_0\mathbf{Y}, \tag{4}$$

where

$$\mathbf{X} = [1 \quad (x-c) \quad (x-c)^2 \quad \ldots \quad (x-c)^N],$$

$$\mathbf{M}_0 = \begin{bmatrix} \frac{1}{0!} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1!} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2!} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{N!} \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} y^{(0)}(c) \\ y^{(1)}(c) \\ y^{(2)}(c) \\ \vdots \\ y^{(N)}(c) \end{bmatrix}.$$

Now we consider the differential part $P_k(x)y^{(k)}(x)$ of Eq. (1) and can write it as the truncated Taylor series expansion of degree N at $x = c$ in the form

$$P_k(x)y^{(k)}(x) = \sum_{n=0}^{N} \frac{1}{n!}[P_k(x)y^{(k)}(x)]^{(n)}_{x=c}(x-c)^n. \tag{5}$$

By the Leibnitz's rule we evaluate

$$[P_k(x)y^{(k)}(x)]^{(n)}_{x=c} = \sum_{i=0}^{n} \binom{n}{i} P_k^{(n-i)}(c)y^{(i+k)}(c)$$

and substitute in expression (5). Thus expression (5) becomes

$$P_k(x)y^{(k)}(x) = \sum_{n=0}^{N}\sum_{i=0}^{n} \frac{1}{(n-i)!i!} P_k^{(n-i)}(c)y^{(i+k)}(c)(x-c)^n \tag{6}$$

and its matrix form

$$[P_k(x)y^{(k)}(x)] = \mathbf{X}\mathbf{P}_k\mathbf{Y}, \tag{7}$$

where

$$
P_k=\begin{bmatrix}
0 & \cdots & 0 & \frac{P_k^{(0)}(c)}{0!0!} & 0 & 0 & \cdots & 0 & 0\\
0 & \cdots & 0 & \frac{P_k^{(1)}(c)}{1!0!} & \frac{P_k^{(0)}(c)}{0!1!} & 0 & \cdots & 0 & 0\\
0 & \cdots & 0 & \frac{P_k^{(2)}(c)}{2!0!} & \frac{P_k^{(1)}(c)}{1!1!} & \frac{P_k^{(0)}(c)}{0!2!} & \cdots & 0 & 0\\
\vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\
0 & \cdots & 0 & \frac{P_k^{(N-k)}(c)}{(N-k)!0!} & \frac{P_k^{(N-k-1)}(c)}{(N-k-1)!1!} & \frac{P_k^{(N-k-2)}(c)}{(N-k-2)!2!} & \cdots & \frac{P_k^{(1)}(c)}{1!(N-k-1)!} & \frac{P_k^{(0)}(c)}{0!(N-k)!}\\
0 & \cdots & 0 & \frac{P_k^{(N-k+1)}(c)}{(N-k+1)!0!} & \frac{P_k^{(N-k)}(c)}{(N-k)!1!} & \frac{P_k^{(N-k-1)}(c)}{(N-k-1)!2!} & & \frac{P_k^{(2)}(c)}{2!(N-k-1)!} & \frac{P_k^{(1)}(c)}{0!(N-k)!}\\
\vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\
0 & \cdots & 0 & \frac{P_k^{(N-1)}(c)}{(N-1)!0!} & \frac{P_k^{(N-2)}(c)}{(N-2)!1!} & \frac{P_k^{(N-3)}(c)}{(N-3)!2!} & \cdots & \frac{P_k^{(k)}(c)}{k!(N-k-1)!} & \frac{P_k^{(k-1)}(c)}{(k-1)!(N-k)!}\\
0 & \cdots & 0 & \frac{P_k^{(N)}(c)}{N!0!} & \frac{P_k^{(N-1)}(c)}{(N-1)!1!} & \frac{P_k^{(N-2)}(c)}{(N-2)!2!} & \cdots & \frac{P_k^{(k+1)}(c)}{(k+1)!(N-k-1)!} & \frac{P_k^{(k)}(c)}{k!(N-k)!}
\end{bmatrix}_{(N+1)x(N+1)}.
$$

Now in a similar way we consider the difference part $P_r^*(x)y^{(r)}(x-\tau)$ of Eq. (1) and can write it as the truncated series expansion of degree N at $x=c$ in the form

$$P_r^*(x)y^{(r)}(x-\tau)=\sum_{n=0}^{N}\frac{1}{n!}[P_r^*(x)y^{(r)}(x-\tau)]_{x=c}^{(n)}(x-c)^n. \tag{8}$$

By the Leibnitz's rule we evaluate

$$[P_r^*(x)y^{(r)}(x-\tau)]_{x=c}^{(n)}=\sum_{i=0}^{n}\binom{n}{i}P_r^{*(n-i)}(c)y^{(i+r)}(c-\tau)$$

and substitute in expression (8). Thus expression (8) becomes

$$P_r^*(x)y^{(r)}(x-\tau)=\sum_{n=0}^{N}\sum_{i=0}^{n}\frac{1}{(n-i)!i!}P_r^{*(n-i)}(c)y^{(i+r)}(c-\tau)(x-c)^n$$

and its matrix form

$$\begin{aligned}&[P_r^*(x)y^{(r)}(x-\tau)]=\mathbf{X}\mathbf{P}_r^*\mathbf{Y}_\tau,\\ &\mathbf{Y}_\tau=[y^{(0)}(c-\tau)\quad y^{(1)}(c-\tau)\quad \ldots\quad y^{(N)}(c-\tau)]^{\mathrm{T}},\end{aligned} \tag{9}$$

where $\mathbf{P}_r^*$ can be obtained by substituting the quantities $P_r^{*(r)}(c)$ instead of $P_k^{(k)}(c)$ in relation (7).

Now substituting quantities $(x-\tau)$ instead of x in (3) and differentiating both side with respect to x we obtain

$$
\begin{aligned}
y^{(0)}(x-\tau) &= \sum_{n=0}^{N} \frac{y^{(n)}(c)}{n!}(x-\tau-c)^{n},\\
y^{(1)}(x-\tau) &= \sum_{n=1}^{N} \frac{y^{(n)}(c)}{(n-1)!}(x-\tau-c)^{n-1},\\
y^{(2)}(x-\tau) &= \sum_{n=2}^{N} \frac{y^{(n)}(c)}{(n-2)!}(x-\tau-c)^{n-2},\\
&\vdots\\
y^{(N)}(x-\tau) &= \sum_{n=N}^{N} \frac{y^{(n)}(c)}{(n-N)!}(x-\tau-c)^{n-N}
\end{aligned}
\tag{10}
$$

or the matrix form for $x=c$

$$\mathbf{Y}_\tau = \mathbf{X}_\tau \mathbf{Y}, \tag{11}$$

where

$$
\mathbf{X}_\tau = \begin{bmatrix}
\frac{1}{0!} & \frac{(-\tau)^1}{1!} & \frac{(-\tau)^2}{2!} & \cdots & \frac{(-\tau)^N}{N!}\\
0 & \frac{1}{0!} & \frac{(-\tau)^1}{1!} & \cdots & \frac{(-\tau)^{N-1}}{(N-1)!}\\
0 & 0 & \frac{1}{0!} & \cdots & \frac{(-\tau)^{N-2}}{(N-2)!}\\
\vdots & \vdots & \vdots & & \vdots\\
0 & 0 & 0 & \cdots & \frac{1}{0!}
\end{bmatrix}_{(N+1)x(N+1)}.
$$

Putting relation (11) in (9), the matrix representation becomes

$$[P_r^*(x)y^{(r)}(x-\tau)] = \mathbf{X}\mathbf{P}_r^*\mathbf{X}_\tau\mathbf{Y}. \tag{12}$$

Let the function $f(x)$ be approximated by a truncated Taylor series

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

Then we can put this series in the matrix form

$$[f(x)] = \mathbf{X}\mathbf{M}_0\mathbf{F}, \tag{13}$$

where

$$\mathbf{F} = [\, f^{(0)}(c) \quad f^{(1)}(c) \quad \ldots \quad f^{(N)}(c)\,]^{\mathrm{T}}.$$

Substituting the matrix forms (7), (12) and (13) corresponding to the functions $P_k(x)y^{(k)}(x)$, $P_r^*(x)y^{(r)}(x-\tau)$ and $f(x)$, into Eq. (1), and then simplifying the resulting equation, we have the matrix equation

$$\left(\sum_{k=0}^{m} \mathbf{P}_k + \sum_{r=0}^{R} \mathbf{P}_r^* \mathbf{X}_\tau\right)\mathbf{Y} = \mathbf{M}_0\mathbf{F}. \tag{14}$$

The matrix equation (14) is a fundamental relation for mth-order linear differential-difference equation with variable coefficients (1).

On the other hand, if we take $(+\tau)$ instead of $(-\tau)$ in Eq. (1) we can obtain the fundamental relation, as (14), of the equation

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) + \sum_{r=0}^{R} P_r^*(x)y^{(r)}(x+\tau) = f(x), \quad \tau > 0. \tag{15}$$

Next, we can obtain the corresponding matrix forms for conditions (2) as follows.

Using relation (10), we find the matrix representations of the functions in (2), for the points a, b and c, in the forms

$$[y^{(k)}(a)] = \mathbf{P}\mathbf{M}_k\mathbf{Y}, \tag{16}$$

$$[y^{(k)}(b)] = \mathbf{Q}\mathbf{M}_k\mathbf{Y}, \tag{17}$$

$$[y^{(k)}(c)] = \mathbf{R}\mathbf{M}_k\mathbf{Y}, \tag{18}$$

where

$$\mathbf{P} = [\,1 \quad (a-c) \quad (a-c)^2 \quad (a-c)^3 \quad \ldots \quad (a-c)^N\,],$$
$$\mathbf{Q} = [\,1 \quad (b-c) \quad (b-c)^2 \quad (b-c)^3 \quad \ldots \quad (b-c)^N\,],$$
$$\mathbf{R} = [\,1 \quad 0 \quad 0 \quad \ldots \quad 0\,],$$

$$M_k = \begin{bmatrix} 0 & 0 & . & . & . & \frac{1}{0!} & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 & \frac{1}{1!} & . & . & . & 0 \\ . & . & & & & . & . & & & & . \\ . & . & & & & . & . & & & & . \\ . & . & & & & . & . & & & & . \\ 0 & 0 & . & . & . & 0 & 0 & . & . & . & \frac{1}{(N-k)!} \\ 0 & 0 & . & . & . & 0 & 0 & . & . & . & 0 \\ . & . & & & & . & . & & & & . \\ . & . & & & & . & . & & & & . \\ . & . & & & & . & . & & & & . \\ 0 & 0 & . & . & . & 0 & 0 & . & . & . & 0 \end{bmatrix}_{(N+1)x(N+1)}.$$

Substituting the matrix representations (16)–(18) into Eq. (2), we obtain the matrices system

$$\sum_{k=0}^{m-1} \{a_{ik}\mathbf{P} + b_{ik}\mathbf{Q} + c_{ik}\mathbf{R}\}\mathbf{M}_k\mathbf{Y} = [\mu_i]. \tag{19}$$

Let us define $\mathbf{U}_i$ as

$$\mathbf{U}_i = \sum_{k=0}^{m-1} \{a_{ik}\mathbf{P} + b_{ik}\mathbf{Q} + c_{ik}\mathbf{R}\}\mathbf{M}_k \equiv [u_{i0} \quad u_{i1} \quad \ldots \quad u_{iN}], \quad i = 0(1)m-1. \tag{20}$$

Thus, the matrix forms of conditions (2) become

$$\mathbf{U}_i\mathbf{Y} = [\mu_i], \quad i = 0, 1, \ldots, m-1. \tag{21}$$

3. Method of solution

Let us consider the fundamental matrix equation (14) corresponding to the mth-order linear differential-difference equation with variable coefficients (1). We can write Eq. (14) in the form

$$\mathbf{WY} = \mathbf{M}_0\mathbf{F}, \tag{22}$$

where

$$\mathbf{W} = [w_{ij}] = \left(\sum_{k=0}^{m} \mathbf{P}_k + \sum_{r=0}^{R} \mathbf{P}_r^*\mathbf{X}_\tau\right), \quad i = 0(1)N, \; j = 0(1)N. \tag{23}$$

The augmented matrix of Eq. (22) becomes

$$[\mathbf{W};\mathbf{M}_0\mathbf{F}]=\begin{bmatrix} w_{00} & w_{01} & . & . & . & w_{0N} & ; & \dfrac{f^{(0)}(c)}{0!} \\ w_{10} & w_{11} & . & . & . & w_{1N} & ; & \dfrac{f^{(1)}(c)}{1!} \\ . & . & & & & . & & \\ . & . & & & & . & & \\ . & . & & & & . & & \\ w_{N0} & w_{N1} & . & . & . & w_{NN} & ; & \dfrac{f^{(N)}(c)}{N!} \end{bmatrix}. \tag{24}$$

We now consider the matrix equations (21) corresponding to conditions (2). Then the augmented matrices of Eqs. (21) become

$$[\mathbf{U}_i;\mu_i]=[u_{i0}\quad u_{i1}\quad \ldots \quad u_{iN}\quad ;\quad \mu_i],\quad i=0(1)(m-1), \tag{25}$$

where the elements $u_{i0}, u_{i1}, \ldots, u_{iN}$ are defined in relation (20).

Consequently, to find the unknown Taylor coefficients $y^{(n)}(c)$, $n=0(1)N$, related with the approximate solution of the problem consisting of Eq. (1) and conditions (2), by replacing the m row matrices (25) by the last m rows of augmented matrix (24), we have new augmented matrix

$$[\mathbf{W}^{*};\mathbf{F}^{*}]=\begin{bmatrix} w_{00} & w_{01} & . & . & . & w_{0N} & ; & \dfrac{f^{(0)}(c)}{0!} \\ w_{10} & w_{11} & . & . & . & w_{1N} & ; & \dfrac{f^{(1)}(c)}{1!} \\ \ldots & \ldots & & & & \ldots & ; & \ldots \\ w_{N-m,0} & w_{N-m,1} & . & . & . & w_{N-m,N} & ; & \dfrac{f^{(N-m)}(c)}{(N-m)!} \\ u_{00} & u_{01} & . & . & . & u_{0N} & ; & \mu_0 \\ u_{10} & u_{11} & . & . & . & u_{1N} & ; & \mu_1 \\ \ldots & \ldots & & & & \ldots & ; & \ldots \\ u_{m-1,0} & u_{m-1,1} & . & . & . & u_{m-1,N} & ; & \mu_{m-1} \end{bmatrix}$$

or the corresponding matrix equation

$$\mathbf{W}^{*}\mathbf{Y}=\mathbf{F}^{*} \tag{26}$$

so that

$$\mathbf{W}^* = \begin{bmatrix} w_{00} & w_{01} & . & . & . & w_{0N} \\ w_{10} & w_{11} & . & . & . & w_{1N} \\ \cdots & \cdots & & & & \cdots \\ w_{N-m,0} & w_{N-m,1} & . & . & . & w_{N-m,N} \\ u_{00} & u_{01} & . & . & . & u_{0N} \\ u_{10} & u_{11} & . & . & . & u_{1N} \\ \cdots & \cdots & . & . & . & \cdots \\ u_{m-1,0} & u_{m-1,1} & . & . & . & u_{m-1,N} \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} y^{(0)}(c) \\ y^{(1)}(c) \\ y^{(2)}(c) \\ . \\ . \\ . \\ y^{(N)}(c) \end{bmatrix},$$

$$\mathrm{F}^* = \begin{bmatrix} \frac{f^{(0)}(c)}{0!} & \frac{f^{(1)}(c)}{1!} & . & . & . & \frac{f^{(N-m)}(c)}{(N-m)!} & \mu_0 & \mu_1 & . & . & . & \mu_{m-1} \end{bmatrix}^{\mathrm{T}}.$$

If $\det \mathbf{W}^* \neq 0$, we can write Eq. (26) as

$$\mathbf{Y} = (\mathbf{W}^*)^{-1}\mathbf{F}^*$$

and the matrix **Y** is uniquely determined. Thus the *m*th-order linear differential-difference equation with variable coefficients (1) with conditions (2) has a unique solution.This solution is given by the truncated Taylor series

$$y(x) = \sum_{n=0}^{N} \frac{y^{(n)}(c)}{n!}(x-c)^n. \tag{27}$$

In the augmented matrix [$\mathbf{W}^*$;$\mathbf{F}^*$], if we take $u_{ij} = 0$ and $\mu_i = 0$, we may obtain the general solution of Eq. (1). In the augmented matrix [$\mathbf{W}$;$\mathbf{M}_0\mathbf{F}$], if $\det \mathbf{W} \neq 0$, we may obtain the particular solution of Eq. (1).

We can easily check the accuracy of this solution as follows:

Since the Taylor polynomial (3) is an approximate solution of Eq. (1), when the solution $y(x)$ and its derivatives are substituted in Eq. (1), the resulting equation must be satisfied approximately; that is, for $x = x_i \in [a, b]$

$$E(x_i) = \left| \sum_{k=0}^{m} P_k(x_i) y^{(k)}(x_i) + \sum_{r=0}^{R} P_r^*(x_i) y^{(r)}(x_i - \tau) - f(x_i) \right| \cong 0$$

or

$$E(x_i) \leqslant 10^{-k_i} \quad (k_i \text{ is any positive integer}).$$

If $\max\ (10^{-k_i}) = 10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $E(x_i)$ at each of the points becomes smaller than the prescribed 10^{-k}.

4. Numerical experiment

In this section, we report on numerical results of some examples, selected differential-difference equations, solved by matrix method described in this paper. For Examples 1–4, we have reported in Tables 1–5, the values of exact solution $y(x)$, polynomial approximate solution $y_N(x)$, absolute error $|y(x) - y_N(x)|$ and estimation error (denoted by exact, Present, e_N, E_N, respectively) at selected points of the given interval.

Example 1. Consider the following third-order linear differential-difference equation with constant coefficients:

$$y'''(x) - y''(x) - y(x) + (e-2)y''(x-1) + y'(x-1) + y(x-1) = 2e - 7$$

with conditions

$$y(0) = 1, \;\; y'(0) = 0, \;\; y''(0) = 1, -1 \leqslant x \leqslant 0$$

and approximate the solution $y(x)$ by the Taylor polynomial

$$y(x) = \sum_{n=0}^{5} \frac{1}{n!} y^{(n)}(c)x(x-c)^n, \tag{28}$$

Table 1
Numerical results of Example 1

N	x	Exact	Present method	e_N	E_N
$N=5$	0.0	1.000000	1.000000	0.000000	0.300E-9
	−0.2	1.021269	1.021385	0.116E-3	0.122E-2
	−0.4	1.089679	1.090666	0.986E-3	0.983E-2
	−0.6	1.211188	1.214754	0.356E-2	0.331E-1
	−0.8	1.390671	1.399830	0.915E-2	0.786E-1
	−1.0	1.632120	1.651724	0.196E-1	0.153660
$N=7$	0.0	1.000000	1.000000	0.000000	0.586E-2
	−0.2	1.021269	1.021294	0.250E-4	0.397E-2
	−0.4	1.089679	1.089903	0.223E-4	0.200E-2
	−0.6	1.211188	1.212030	0.842E-4	0.384E-3
	−0.8	1.390671	1.392907	0.223E-2	0.412E-2
	−1.0	1.632120	1.637029	0.490E-2	0.109E-2
$N=9$	0.0	1.000000	1.000000	0.000000	0.658E-2
	−0.2	1.021269	1.021269	0.476E-6	0.518E-2
	−0.4	1.089679	1.089695	0.152E-4	0.393E-2
	−0.6	1.211188	1.211279	0.909E-4	0.276E-2
	−0.8	1.390671	1.390985	0.314E-3	0.158E-2
	−1.0	1.632120	1.632941	0.821E-3	0.223E-3

Table 2
Numerical results of Example 2

N	x	Exact	Present method	e_N	E_N
$N=4$	0.0	1.000000	1.000000	0.000000	0.400E-8
	−0.2	0.677461	0.677554	0.929E-4	0.751E-2
	−0.4	0.500640	0.499863	0.776E-3	0.557E-1
	−0.6	0.457623	0.453034	0.458E-2	0.173502
	−0.8	0.538657	0.525359	0.132E-1	0.376299
	−1.0	0.735758	0.707317	0.284E-1	0.666666
$N=6$	0.0	1.000000	1.000000	0.000000	0.400E-8
	−0.2	0.677461	0.677518	0.565E-4	0.104E-4
	−0.4	0.500640	0.500954	0.314E-3	0.304E-3
	−0.6	0.457623	0.458517	0.894E-3	0.210E-2
	−0.8	0.538657	0.540542	0.188E-2	0.794E-2
	−1.0	0.735758	0.739080	0.332E-2	0.214E-1
$N=8$	0.0	1.000000	1.000000	0.000000	0.400E-8
	−0.2	0.677461	0.677453	0.817E-5	0.880E-8
	−0.4	0.500640	0.500599	0.406E-4	0.144E-5
	−0.6	0.457623	0.457516	0.107E-3	0.235E-4
	−0.8	0.538657	0.538445	0.212E-3	0.166E-3
	−1.0	0.735758	0.735403	0.355E-3	0.749E-3

Table 3
Numerical results of Example 3

N	x	Exact	Present method	e_N	E_N
$N=9$	0.0	1.000000	1.000000	0.000000	0.000000
	−0.2	1.221402	1.221251	0.123E-3	0.400E-8
	−0.4	1.491824	1.491179	0.432E-3	0.512E-6
	−0.6	1.822118	1.820567	0.851E-2	0.856E-5
	−1.0	2.718281	2.713298	0.183E-2	0.288E-3
	−2.0	7.389056	7.353509	0.481E-2	0.254E-1
$N=10$	0.0	1.000000	1.000000	0.000000	0.200E-9
	−0.2	1.221402	1.221521	0.118E-3	0.100E-8
	−0.4	1.491824	1.492332	0.508E-3	0.000000
	−0.6	1.822118	1.823340	0.122E-2	0.103E-6
	−1.0	2.718281	2.722206	0.392E-2	0.204E-4
	−2.0	7.389056	7.417016	0.279E-1	0.117E-1
$N=11$	0.0	1.000000	1.000000	0.000000	0.000000
	−0.2	1.221402	1.221458	0.554E-4	0.000000
	−0.4	1.491824	1.492061	0.237E-3	0.600E-8
	−0.6	1.822118	1.822688	0.569E-3	0.202E-6
	−1.0	2.718281	2.720112	0.183E-2	0.187E-4
	−2.0	7.389056	7.402098	0.130E-1	0.783E-2

Table 4
The maximum error for $\varepsilon = 1$ and for different δ values

δ	$e_N = 5$	$e_N = 6$	$e_N = 7$
0.01	0.005766	0.001419	0.000291
0.03	0.006249	0.001616	0.000286
0.06	0.007000	0.002052	0.000194

Table 5
The maximum error for $\varepsilon = 2$ and for different δ values

δ	$e_N = 5$	$e_N = 6$	$e_N = 7$
0.01	0.0003259	0.0000478	0.0000054
0.03	0.0003328	0.0000558	0.5040E-7
0.06	0.0003328	0.0000558	0.0000206

where $N = 5$, $c = 0$, $\tau = 1$, $P_0 = -1$, $P_2(x) = -1$, $P_3(x) = 1$, $P_0^*(x) = 1$, $P_1^*(x) = 1$, $P_2^*(x) = (e - 2)$, $f(x) = 2e - 7$.

Then, for $N = 5$, the matrix equation (14) becomes

$$[\mathbf{P}_0 + \mathbf{P}_2 + \mathbf{P}_3 + (\mathbf{P}_1^* + \mathbf{P}_2^* + \mathbf{P}_3^*)\mathbf{X}_1]\mathbf{Y} = \mathbf{M}_0\mathbf{F},$$

where

$$P_0 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{24} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{120} \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_0^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{24} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{120} \end{bmatrix},$$

$$P_1^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{24} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_2^* = \begin{bmatrix} 0 & 0 & e-2 & 0 & 0 & 0 \\ 0 & 0 & 0 & e-2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}e-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6}e-\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$X_1 = \begin{bmatrix} 1 & -1 & \frac{1}{2} & \frac{-1}{6} & \frac{1}{24} & \frac{-1}{120} \\ 0 & 1 & -1 & \frac{1}{2} & \frac{-1}{6} & \frac{1}{24} \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{-1}{6} \\ 0 & 0 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{24} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{120} \end{bmatrix},$$

$$\mathbf{F} = [-1.563436344 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}},$$

$$\mathbf{Y} = [y^{(0)}(0) \quad y^{(1)}(0) \quad y^{(2)}(0) \quad y^{(3)}(0) \quad y^{(4)}(0) \quad y^{(5)}(0)]^{\mathrm{T}}.$$

For the conditions $y(0)=1,\ y'(0)=0$ and $y''(0)=1$, the augmented matrices become

$$\begin{aligned} [\mathbf{U}_0;\mu_0] &= [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad ; \quad 1], \\ [\mathbf{U}_1;\mu_1] &= [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad ; \quad 0], \\ [\mathbf{U}_2;\mu_2] &= [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad ; \quad 1] \end{aligned}$$

from (25). Using the matrices $\mathbf{P}_0,\ \mathbf{P}_2,\ \mathbf{P}_3,\ \mathbf{P}_0^*,\ \mathbf{P}_1^*,\ \mathbf{P}_2^*,\ \mathbf{X}_1,\ \mathbf{M}_0$, and $\mathbf{F}$, we find matrices $\mathbf{W}^*$ and $\mathbf{F}^*$ in (26) as

$$\mathbf{W}^* = \begin{bmatrix} 0 & 0 & \frac{-7}{2}+e & \frac{10}{3}-e & \frac{-9}{8}+\frac{1}{2}e & \frac{11}{30}-\frac{1}{6}e \\ 0 & 0 & 0 & \frac{-7}{2}+e & \frac{10}{3}-e & \frac{-9}{8}+\frac{1}{2}e \\ 0 & 0 & 0 & 0 & \frac{-7}{4}+\frac{1}{2}e & \frac{5}{3}-\frac{1}{2}e \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{F}^* = [-1.563436344 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1]$$

and thus the solution

$$\mathbf{Y} = [1 \quad 0 \quad 1 \quad -1.083363644 \quad -0.9279509059 \quad -1.179407056]^{\mathrm{T}}. \tag{29}$$

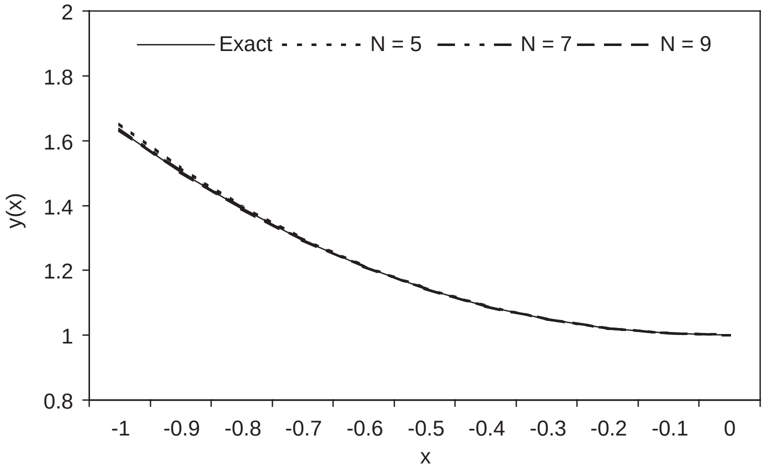

Fig. 1. Numerical and exact solution of Example 1 for various N.

Substituting the elements of column matrix (29) into (28), we obtain the approximate solution in terms of the Taylor polynomial of degree five about $x = 0$ as

$$y(x) = 1 + 0.5x^2 - 0.1805606073x^3 - 0.03866462108x^4 - 0.009828392133x^5.$$

We use the absolute error to measure the difference between the numerical and exact solutions. In Table 1 the solutions obtained for $N = 5, 7, 9$ are compared with the exact solution $y(x) = x^2 + x + 2 - e^x$ [6] (see Fig. 1).

Example 2. Secondly we can take the problem

$$2y''(x) + 2y'(x) - 4y(x) + y''(x-1) + y'(x-1) - 2y(x-1) = -6x^2 + 10x + 8,$$

$$y(0) = 1, \quad y'(0) = 2$$

so that $N = 4$, $c = 0$, $\tau = 1$, $P_0(x) = -4$, $P_1(x) = 2$, $P_2(x) = 2$, $P_0^*(x) = -2$, $P_1^*(x) = 1$, $P_2^*(x) = 1$, $f(x) = -6x^2 + 10x + 8$. Then for $N = 4$, the matrix equation is obtained as

$$[\mathbf{P}_0 + \mathbf{P}_1 + \mathbf{P}_2 + (\mathbf{P}_0^* + \mathbf{P}_1^* + \mathbf{P}_2^*)\mathbf{X}_1]\mathbf{Y} = \mathbf{M}_0\mathbf{F}.$$

Following the previous procedures, we find matrices $\mathbf{W}^*$ and $\mathbf{F}^*$ in (26) as

$$W^* = \begin{bmatrix} -6 & 5 & 1 & \frac{-1}{6} & \frac{1}{4} \\ 0 & -6 & 5 & 1 & \frac{-1}{6} \\ 0 & 0 & -3 & \frac{5}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$F^* = [8 \quad 10 \quad -6 \quad 1 \quad 2]^{\mathrm{T}}$$

and its solution

$$Y = [1 \quad 2 \quad \tfrac{494}{123} \quad \tfrac{88}{41} \quad \tfrac{56}{41}]. \tag{30}$$

By using the elements of column matrix (30), we obtain the solution

$$y(x) = 1 + 2x + \tfrac{247}{123}x^2 + \tfrac{44}{123}x^3 + \tfrac{7}{123}x^4.$$

The solutions obtained for $N = 4, 6, 8$ are compared with the exact solution $y(x) = 2e^x + x^2 - 1$, which are given in Table 2.

Example 3. We now consider the equation with variable coefficients

$$y''(x) + xy'(x) + xy(x) + y'(x-1) + y(x-1) = e^{-x},$$

$$y(0) = 1, \quad y'(0) = -1.$$

The exact solution is $y(x) = e^{-x}$. For numerical results see Table 3.

Example 4 (*Kadalbajoo and Sharma [4, Example 1]*).

$$\varepsilon y''(x) + y'(x-\delta) - y(x) = 0, \quad -\delta \leqslant x \leqslant 0,$$

$$y(0) = 1, \quad y(1) = 1.$$

The exact solution is

$$y(x) = \frac{(1 - e^{m_2})e^{m_1 x} + (e^{m_1} - 1)e^{m_2 x}}{e^{m_1} - e^{m_2}},$$

where

$$m_1 = \frac{-1 - \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)}, \quad m_2 = \frac{-1 + \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)}.$$

For numerical results, see Tables 4 and 5.

5. Conclusions

High-order linear differential-difference equations with variable coefficients are usually difficult to solve analytically. In many cases, it is required approximate solutions. The present method is based on computing the coefficients in the Taylor expansion of solution of a linear differential-difference equation.

To get the best approximating solution of the equation, we take more terms from the Taylor expansion of functions; that is, the truncation limit N must be chosen large enough. From the tabular points shown in Table 1, it may be observed that the solution found for $N = 7$ shows close agreement for various values of x_i. In particular, the solution of Example 2 for $N = 8$ shows a very close approximation to the exact solution at the points in interval $-1 \leqslant x \leqslant 0$.

In Example 4, we compare the results for different ε and δ values and it is seen that when $\varepsilon \geqslant 1$, our results are in a good agreement with the exact solution. Also, if $0 < \varepsilon < 1$, some modifications are required.

If the Taylor polynomial solutions are looked for about the points in the given conditions, we see that there exists a solution which is closer to the exact solution. In the matrix $[\mathbf{W};\mathbf{M}_0\mathbf{F}]$, if $\det\mathbf{W} \neq 0$, we can obtain a particular solution of Eq. (1). If the conditions given in (2) are not used and if $\mathrm{rank}[\mathbf{W}] = \mathrm{rank}[\mathbf{W};\mathbf{M}_0\mathbf{F}] = N + 1$ in (22), then by replacing the last m rows of the matrix $[\mathbf{W};\mathbf{M}_0\mathbf{F}]$ with zero, the general solution may be obtained.

A considerable advantage of the method is that Taylor coefficients of the solution are found very easily by using the computer programs. We use the symbolic algebra program, Maple, to find the Taylor coefficients of the solution.

The method can be developed and applied to system of linear difference equations. Also, the method may be used to solve integrodifferential-difference equations in the form

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) + \sum_{h=1}^{n} P_h^*(x)y^{(h)}(x-h) = f(x) + \lambda\int_a^b \sum_{i=0}^{p} K_i(x,t)y^{(i)}(t)\,\mathrm{d}t + \mu\int_a^x \sum_{j=0}^{q} K_j(x,t)y^{(j)}(t)\,\mathrm{d}t$$

but some modifications are required. Note that the presented method can be used for solving the differential-difference equations with positive shift, too.

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References

[1] D.D. Bainov, M.B. Dimitrova, A.B. Dishliev, Oscillation of the bounded solutions of impulsive differential-difference equations of second order, Appl. Math. Comput. 114 (2000) 61–68.
[2] J. Cao, J. Wang, Delay-dependent robust stability of uncertain nonlinear systems with time delay, Appl. Math. Comput. 154 (2004) 289–297.
[3] M.K. Kadalbajoo, K.K. Sharma, Numerical analysis of boundary-value problems for singularly-perturbed differential-difference equations with small shifts of mixed type, J. Optim. Theory Appl. 115 (2002) 145–163.
[4] M.K. Kadalbajoo, K.K. Sharma, Numerical analysis of singularly perturbed delay differential equations with layer behavior, Appl. Math. Comput. 157 (2004) 11–28.
[5] K.P. Kanwal, K.C. Liu, A Taylor expansion approach for solving integral equation, Internat. J. Math. Ed. Sci. Tech. 20 (3) (1989) 411–414.
[6] H. Levy, F. Lesman, Finite Difference Equations, The Macmillan Company, NewYork, 1961.
[7] S. Nas, S. Yalçinbas, M. Sezer, A Taylor polynomial approach for solving high-order linear Fredholm integro-differential equations, Internat. J. Math. Ed. Sci. Technol. 31 (2) (2000) 213–225.
[8] T.L. Saaty, Modern Nonlinear Equations, Dover Publications, Inc., NewYork, 1981.
[9] M. Sezer, Taylor polynomial solution of Volterra Integral equations, Internat. J. Math. Ed. Sci. Technol. 25 (5) (1994) 625–633.
[10] M. Sezer, A method for approximate solution of the second order linear differential equations in terms of Taylor polynomials, Internat. J. Math. Ed. Sci. Technol. 27 (6) (1996) 821–834.
[11] H. Zuoshang, Boundness of solutions to functional integro-differential equations, Proc. Amer. Math. Soc. 114 (2) (1992).