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Dynamic aggregation operators and Einstein operations based on interval-valued picture hesitant fuzzy information and their applications in multi-period decision making

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Abstract

The traditional picture hesitant fuzzy aggregation operators are generally suitable for aggregating information acquired in the form of picture hesitant fuzzy numbers, but they will fail in dealing with interval-valued picture hesitant fuzzy information. In this paper, we describe the notion of interval-valued picture hesitant fuzzy set and the operational laws of interval-valued picture hesitant fuzzy variables. Moreover, we derive some dynamic interval-valued picture hesitant fuzzy aggregation operators (based on Einstein operators) to aggregate the interval-valued picture hesitant fuzzy information collected at different periods. Some desirable properties of these aggregation operators are discussed in detail. In addition, we develop the approaches to tackle the multi-period decision-making problems, where all decision information takes the form of interval-valued picture hesitant fuzzy information collected at different periods. In an attempt to illustrate the applications of the proposed approaches, two numerical examples are given to measure the impact of Coronavirus Disease 2019 (COVID-19) in daily life and to identify the optimal investment opportunity. Finally, a comparative analysis of the proposed and existing studies are conducted to demonstrate the effectiveness of the proposed approaches. The presented interval-valued picture hesitant fuzzy operations, aggregation operators, and decision-making approaches can widely apply to dynamic decision analysis and multi-stage decision analysis in real life.

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1 Introduction

In 1965, Zadeh (1965) introduced the idea of fuzzy logic which is the expansion of binary logic. When handling a few involute systems, due to the flexibility of membership of fuzzy logic, many generalized styles of the fuzzy set (FS) and fuzzy element (FE) were developed in the following year. Some of these are as follows: interval-valued fuzzy set (IVFS) Sambuc (1975), in which value of membership in the fuzzy set is interval; intuitionistic fuzzy set (IFS) Atanassov (1986, 1989), which is created by integrating the function of non-membership into the form of a fuzzy set; interval-valued intuitionistic fuzzy set (IVIFS) Atanassov and Gargov (1989), in which values of membership and non-membership in the intuitionistic fuzzy set are intervals. These paradigms were studied by many researchers in various aspects (Kamacı 2019; Karaaslan and Karataş 2016; Khan and Zhu 2020; Khan et al. 2019a; Petchimuthu et al. 2020; Petchimuthu and Kamacı 2020; Riaz and Tehrim 2019, 2020). Many researchers dealt with the extended types of IFSs that stretch the underlying requirement of IFS that the sum of membership degree and non-membership degree is less than or equal to 1 (Naeem et al. 2019; Riaz et al. 2020; Wang et al. 2020; Wang and Li 2020). In 2010, Torra (2010) introduced the notion of a hesitant fuzzy set, which permits the membership to have a set of possible values. This paradigm allows overcoming some situations in which experts hesitate between several possible membership values to assess an alternative. Xia and Xu (2011) defined a few operational regulations for HFS based on the algebraic t-norm and t-conorm, and then advanced several hesitant fuzzy aggregation operators to aggregate hesitant fuzzy information. Many authors (Deli and Karaaslan 2020; Wei 2012; Zhang 2013a; Zhu et al. 2012b) developed different types of aggregation operators to aggregate the possible values of membership of alternatives in the decision making under the surrounding of HFS. Chen et al. (2013) and Wei et al. (2013) generalized the framework of the HFS to the interval-valued hesitant fuzzy set (IVHFS). They argued that the membership degrees in an IVHFS can be both the set of exact fuzzy numbers and the set of interval-valued fuzzy numbers. Jin et al. (2016) advanced the Einstein prioritized aggregation operators for the interval-valued hesitant fuzzy variables. Beg and Rashid (2014) blended the notions of IFS and HFS to create a brand new perception referred to as the intuitionistic hesitant fuzzy set (IHFS). Zhang (2013b) studied the interval-valued intuitionistic hesitant fuzzy set (IVIHFS), which is the generalized form of IHFSs, and their basic properties. Zhu et al. (2012a) proposed the idea of hybridization of a hesitant fuzzy set with duality, and thus initiated a dual hesitant fuzzy set (DHFS) including possible values of membership and non-membership. They investigated some fundamental operations and properties of DHFS. Moreover, Wang et al. (2014) advanced a few aggregation operators based on dual hesitant fuzzy elements (DHFEs), such as the dual hesitant fuzzy weighted average, weighted geometric, ordered weighted average, ordered weighted geometric, hybrid average, and hybrid geometric operators. In Ju et al. (2014b), the authors derived some dual hesitant fuzzy Hamacher average and geometric aggregation operators to apply in a multiple attribute decision-making problem. Ju et al. (2014a) generalized the dual hesitant fuzzy set to interval-valued dual hesitant fuzzy set (IVDHFS) and developed the interval-valued dual hesitant fuzzy aggregation operators. Moreover, Zhang et al. (2014) published a paper on the interval-valued dual hesitant fuzzy Einstein weighted aggregation operators like ordered weighted average operator, ordered weighted geometric operator, hybrid average operator, and hybrid geometric operators. On the other hand, Cuong (2014) proposed integrating the grade of indeterminacy into the structure of IFS and defined a picture fuzzy set (PFS), which is characterized by the degrees of positive membership, neutral membership (or indeterminacy), and negative membership (or non-membership). They pointed out that the simplest

constraint is that the sum of the three degrees must not exceed 1. This constraint allows PFSs to have a different structure than neutrosophic sets (see: Akram et al. 2018; Kamacı 2020; Riaz and Hashmi 2020; Smarandache 1998; Tehrim and Riaz 2019 for neutrosophic sets). In addition, many researchers derived various aggregation operators to aggregate the picture fuzzy aggregation information, e.g., the picture fuzzy aggregation operators and picture fuzzy Hamacher aggregation operators (Wei 2017, 2018), the picture fuzzy aggregation operators based on Einstein operations (Khan et al. 2019b), and the picture fuzzy hybrid averaging operator (Garg 2017). In Khalil et al. (2019), a generalization of the concept of picture fuzzy set was given in the spirit of ordinary interval-valued fuzzy sets, and thus initiated the theory of interval-valued picture fuzzy set (IVPFS). Wang and Li (2018) introduced picture hesitant fuzzy set (PHFS) by combining the PFS and HFS, and proposed the generalized picture hesitant fuzzy weighted aggregation operators and the generalized picture hesitant fuzzy prioritized weighted aggregation operators to solve the multi-criteria decision-making (MCDM) problems.

In general, the above studies were dedicated to proposing models aggregating the hesitant/picture/picture hesitant fuzzy information in the same period. However, some complex decision-making problems have to consider the performance of alternatives at the different periods. Such problems are said to be multi-period decision-making (MPDM) problems, where information is collected in time moments of a period. In recent years, many authors investigated the temporal generalized versions (often called dynamic) of the existing hesitant/picture/picture hesitant fuzzy aggregation operators and discussed their efficiency in the MPDM. For instance, Lin et al. (2008) created a dynamic multi-attribute decision-making model with grey number evaluations, Peng and Wang (2014) proposed dynamic hesitant fuzzy aggregation operators to deal with multi-period decision-making problems. While the current picture hesitant fuzzy aggregation operators are suitable for aggregating the picture hesitant fuzzy information collected at the same period, these aggregation operators cannot be used to deal with MPDM problems under the picture hesitant fuzzy environment.

Based on the above analysis, we find that how to extend the PHFS to interval-valued picture hesitant fuzzy set (IVPHFS), where the degrees of positive membership, neutral membership, and negative membership are the sets of possible fuzzy intervals and how to aggregate the interval-valued picture hesitant fuzzy information collected at different periods are meaningful research issues. This paper is motivated to fill these research gaps in the literature. Now, we discuss some important objectives of this study.

1. In some real-life problems based on the nature of PHFS, the degrees of positive membership, neutral membership, and negative membership may be the sets of possible fuzzy intervals. The PHFS fail in such situations. To overcome these deficiencies, the concept of IVPHFS that extends the intuitive, hesitant, and picture hybridizations of fuzzy sets is defined. The relationship among interval-valued picture hesitant fuzzy set and several existing fuzzy sets is presented in Fig. 1.
   The IVPHFSs allow the membership of objects that cannot be characterized by the existing fuzzy sets (in Fig. 1) to be clearly described. In addition, the basic operations and relations presented for IVPHFSs and IVPHFEs contribute to both theory and practice in fuzzy mathematics.
2. The existing types of picture hesitant fuzzy aggregation operators are suitable for aggregating the picture hesitant fuzzy information collected in the same period. But these existing aggregation operators cannot deal with situations, where information is collected at the different periods. To fuse interval-valued picture hesitant fuzzy information collected at the different periods, the dynamic interval-valued picture hesitant fuzzy aggregation oper-
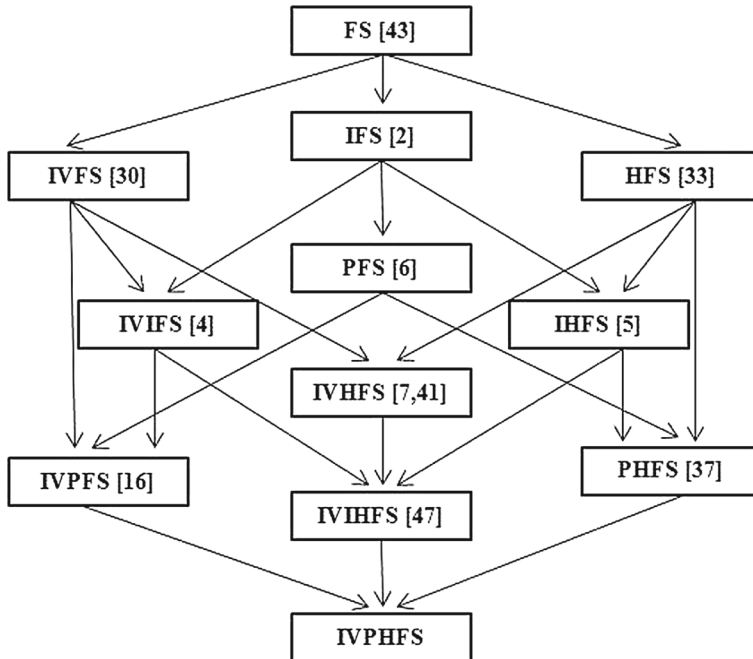
Fig. 1 Relationship among interval-valued picture hesitant fuzzy set and other fuzzy sets

ators (based on Einstein operations) and their generalized forms are proposed. Each of these proposed dynamic aggregation operators proceeds differently during calculations, and so it is put forward that in some cases one can be superior to the other. It is obvious that if it is taken as $t = t_1$ (i.e., the period has only one time moment) in the formulations of dynamic interval-valued picture hesitant fuzzy aggregation operators then these dynamic aggregation operators turn into traditional interval-valued picture hesitant fuzzy aggregation operators. Thus, it is highlighted that the developed dynamic aggregation operators aggregate interval-valued picture hesitant fuzzy information collected both at the same period and the different periods.

3. It is not possible to solve the multi-period decision-making problems using the existing picture hesitant fuzzy arguments. This study proposes the approaches using the developed dynamic interval-valued picture hesitant fuzzy aggregation operators to deal with diverse situations during multi-period decision-making process under the interval-valued picture hesitant fuzzy environment. In addition, it is shown that these approaches can be applied to real-life problems such as measuring the impact of COVID-19 and determining the optimal choice of investment.

The remainder of the paper is prepared as follows: in Sect. 2, the concept of IVPHFS and some interval-valued picture hesitant fuzzy aggregation operators that can be used to fuse time-independent interval-valued picture hesitant fuzzy arguments are introduced. Section 3 describes some new dynamic interval-valued picture hesitant fuzzy weighted aggregation operators and their generalized versions to aggregate interval-valued picture hesitant fuzzy information collected at different periods. In Sect. 4, the notions of (generalized) dynamic interval-valued picture hesitant fuzzy weighted aggregation operators based on Einstein operations and their essential properties such as idempotency, boundedness, and monotonicity

are given. Section 5 proposes two approaches using new dynamic aggregation operators to the MPDM, where all decision information takes the form of interval-valued picture hesitant fuzzy information collected at different periods. The numerical examples are given to illustrate how to apply the developed decision-making approaches. Section 6 is devoted to the comparative analysis, followed by concluding remarks in Sect. 7.

2 Interval-valued picture hesitant fuzzy set

In this part, we introduce the concepts of interval-valued picture hesitant fuzzy set and interval-valued picture hesitant fuzzy element. Moreover, we give some essential notions (like union, intersection, aggregation operators) for interval-valued picture hesitant fuzzy elements.

From now on, $I^{[0,1]}$ denotes the set of all closed subintervals of $[0, 1]$. If $x \in I^{[0,1]}$ then $x = [x^L, x^U]$, such that $x^L \leq x^U$.

Definition 2.1 Let Q be a nonempty and finite set. $I^{[0,1]}$ denotes the set of all closed subintervals of $[0, 1]$. Then, an interval-valued picture hesitant fuzzy set (IVPHFS) $\widetilde{P}$ in Q is defined as

$$\widetilde{P} = \{\langle q, \widetilde{\wp}(q)\rangle : q \in Q\} = \{\langle q, \widetilde{\alpha}(q), \widetilde{\beta}(q), \widetilde{\gamma}(q)\rangle : q \in Q\}$$

where $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}$ are, respectively, the positive membership function, neutral membership function, and negative membership function of the set $\widetilde{P}$, and each of them when applied to Q returns a subset of $I^{[0,1]}$. For each $q \in Q$, $\widetilde{\alpha}(q), \widetilde{\beta}(q)$ and $\widetilde{\gamma}(q)$ are the sets of some closed subintervals of $[0, 1]$, denoting the possible degrees of positive membership, neutral membership, and negative membership of $q \in Q$ to the set $\widetilde{P}$, respectively. That is $\widetilde{\alpha}(q) = \{a : a = [a^L, a^U] \in I^{[0,1]}\}$, $\widetilde{\beta}(q) = \{b : b = [b^L, b^U] \in I^{[0,1]}\}$ and $\widetilde{\gamma}(q) = \{c : c = [c^L, c^U] \in I^{[0,1]}\}$ with the following condition

$$0 \leq a_{\max}^U + b_{\max}^U + c_{\max}^U \leq 1$$

where

$$a_{\max}^U = \bigcup_{[a^L, a^U] \in \widetilde{\alpha}(q)} \max\{a^U\}, \quad b_{\max}^U = \bigcup_{[b^L, b^U] \in \widetilde{\beta}(q)} max\{b^U\} and c_{\max}^U = \bigcup_{[c^L, c^U] \in \widetilde{\gamma}(q)} \max\{c^U\}.$$

For convenience, the triplet $\widetilde{\wp}(q) = \{\widetilde{\alpha}(q), \widetilde{\beta}(q), \widehat{\gamma}(q)\}$ is called an interval-valued picture hesitant fuzzy element (IVPHFE), denoted by

$$\widehat{\wp} = \{\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}\} = \bigcup_{[a^L, a^U] \in \widehat{\alpha}, \, [b^L, b^U] \in \widehat{\beta}, \, [c^L, c^U] \in \widehat{\gamma}} \{\{[a^L, a^U]\}, \{[b^L, b^U]\}, \{[c^L, c^U]\}\}$$

(2.1)

or

$$\widehat{\wp} = \{\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}\} = \bigcup_{[a^L, a^U] \in \widehat{\alpha}, \, [b^L, b^U] \in \widehat{\beta}, \, [c^L, c^U] \in \widehat{\gamma}} \begin{Bmatrix} \{[a^L, a^U]\} \\ \{[b^L, b^U]\} \\ \{[c^L, c^U]\} \end{Bmatrix}.$$

(2.2)

Throughout this paper, the term IVPHFE may sometimes appear as interval-valued picture hesitant fuzzy variable (IVPHFV) or interval-valued picture hesitant fuzzy number (IVPHFN). $\mathcal{P}$ denotes the set of all IVPHFEs (or IVPHFVs/IVPHFNs) on Q.

Example 2.2 Air pollution is due to many reasons such as industrial wastes and fumes, heat and fume of vehicles and machinery, uncontrolled thermal power plants, and irregular heating systems (excessive coal consumption, and so on). Considering the areal and temporal variability of air pollution and the uncertainties that cause pollution, it may not be sufficient to create the membership (or non-membership) degree based on the fuzzy logic or the intuitionistic fuzzy logic for the air pollution of each city. This can be considered as an illustrative example of interval-valued picture hesitant fuzzy logic.

Let $Q = \{q_1 = Ankara, q_2 = Istanbul, q_3 = Yozgat, q_4 = Antalya\}$ be a set of five cities in Turkey. According to "air pollution" of the cities, we can create the IVPHFS as follows:

$$
\widetilde{P} = \left\{
\begin{array}{l}
\langle q_1, \{[0.2, 0.4]\}, \{[0.1, 0.3], [0.2, 0.4]\}, \{[0, 0.1], [0.1, 0.2]\}\rangle, \\
\langle q_2, \{[0.1, 0.3]\}, \{[0.2, 0.2], [0.3, 0.4], [0.2, 0.5]\}, \{[0, 0.1]\}\rangle, \\
\langle q_3, \{[0.2, 0.35], [0.3, 0.4]\}, \{[0.1, 0.25], [0.2, 0.3]\}, \{[0.1, 0.2]\}\rangle, \\
\langle q_4, \{[0.1, 0.3], [0.2, 0.5]\}, \{[0.2, 0.2]\}, \{[0.1, 0.1]\}\rangle
\end{array}
\right\} .
$$

Definition 2.3 Let $\widehat{\wp} = \{\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}\} = \bigcup\limits_{[a^L,a^U] \in \widehat{\alpha}, [b^L,b^U] \in \widehat{\beta}, [c^L,c^U] \in \widehat{\gamma}} \{[[a^L, a^U]\}, \{[b^L, b^U]\},$ $\{[c^L, c^U]\}\}$ be an IVPHFE, the numbers of values in $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}$ are $v(\widehat{\alpha}), v(\widehat{\beta}), v(\widehat{\gamma})$, respectively.

(i) The score function is defined as

$$
S(\widehat{\wp}) = \frac{1 + \frac{1}{2}(\frac{1}{v(\widehat{\alpha})} \sum\limits_{[a^L,a^U] \in \widehat{\alpha}} (a^L + a^U) - \frac{1}{v(\widehat{\beta})} \sum\limits_{[b^L,b^U] \in \widehat{\beta}} (b^L + b^U) - \frac{1}{v(\widehat{\gamma})} \sum\limits_{[c^L,c^U] \in \widehat{\gamma}} (c^L + c^U))}{2},
$$
$$
S(\widehat{\wp}) \in [0, 1] \tag{2.3}
$$

(ii) The accuracy function is defined as

$$
A(\widehat{\wp}) = \frac{1}{2}(\frac{1}{v(\widehat{\alpha})} \sum\limits_{[a^L,a^U] \in \widehat{\alpha}} (a^L + a^U) + \frac{1}{v(\widehat{\beta})} \sum\limits_{[b^L,b^U] \in \widehat{\beta}} (b^L + b^U) + \frac{1}{v(\widehat{\gamma})} \sum\limits_{[c^L,c^U] \in \widehat{\gamma}} (c^L + c^U)),
$$
$$
A(\widehat{\wp}) \in [0, 1] \tag{2.4}
$$

(iii) The certainty function is defined as

$$
C(\widehat{\wp}) = \frac{1 + \frac{1}{2}(\frac{1}{v(\widehat{\alpha})} \sum\limits_{[a^L,a^U] \in \widehat{\alpha}} (a^L + a^U) - \frac{1}{v(\widehat{\gamma})} \sum\limits_{[c^L,c^U] \in \widehat{\gamma}} (c^L + c^U))}{2}, \quad C(\widehat{\wp}) \in [0, 1]
$$
$$
\tag{2.5}
$$

Using the notions of score function, accuracy function and certainty function of IVPHFE, the two IVPHFEs can be compared as follows:

Definition 2.4 Let $\widehat{\wp}_1 = \{\widehat{\alpha}_1, \widehat{\beta}_1, \widehat{\gamma}_1\}$ and $\widehat{\wp}_2 = \{\widehat{\alpha}_2, \widehat{\beta}_2, \widehat{\gamma}_2\}$ be two IVPHFEs, then one can compare them in terms of the following rules:

1. if $S(\widehat{\wp}_1) < S(\widehat{\wp}_2)$ then $\widehat{\wp}_1 \prec \widehat{\wp}_2$,
2. if $S(\widehat{\wp}_1) > S(\widehat{\wp}_2)$ then $\widehat{\wp}_1 \succ \widehat{\wp}_2$,
3. if $S(\widehat{\wp}_1) = S(\widehat{\wp}_2)$ then

(i) if $\mathcal{A}(\widehat{\wp}_1) < \mathcal{A}(\widehat{\wp}_2)$ then $\widehat{\wp}_1 \prec \widehat{\wp}_2$,

(ii) if $\mathcal{A}(\widehat{\wp}_1) > \mathcal{A}(\widehat{\wp}_2)$ then $\widehat{\wp}_1 \succ \widehat{\wp}_2$,

(iii) if $\mathcal{A}(\widehat{\wp}_1) = \mathcal{A}(\widehat{\wp}_2)$ then

    (a) if $\mathcal{C}(\widehat{\wp}_1) < \mathcal{C}(\widehat{\wp}_2)$ then $\widehat{\wp}_1 \prec \widehat{\wp}_2$,

    (b) if $\mathcal{C}(\widehat{\wp}_1) > \mathcal{C}(\widehat{\wp}_2)$ then $\widehat{\wp}_1 \succ \widehat{\wp}_2$,

    (c) if $\mathcal{C}(\widehat{\wp}_1) = \mathcal{C}(\widehat{\wp}_2)$ then $\widehat{\wp}_1 \approx \widehat{\wp}_2$.

In above definition, we can simplify it as a binary relation $\precsim_{(\mathcal{S},\mathcal{A},\mathcal{C})}$ on $\mathcal{P}$ given as

$$\widehat{\wp}_1 \precsim_{(\mathcal{S},\mathcal{A},\mathcal{C})} \widehat{\wp}_2 \Leftrightarrow \left(\begin{array}{c} (\mathcal{S}(\widehat{\wp}_1) < \mathcal{S}(\widehat{\wp}_2)) \vee ((\mathcal{S}(\widehat{\wp}_1) = \mathcal{S}(\widehat{\wp}_2)) \wedge (\mathcal{A}(\widehat{\wp}_1) < \mathcal{A}(\widehat{\wp}_2))) \\ \vee ((\mathcal{S}(\widehat{\wp}_1) = \mathcal{S}(\widehat{\wp}_2)) \wedge (\mathcal{A}(\widehat{\wp}_1) = \mathcal{A}(\widehat{\wp}_2)) \wedge (\mathcal{C}(\widehat{\wp}_1) \leq \mathcal{C}(\widehat{\wp}_2))) \end{array} \right)$$

for all $\widehat{\wp}_1, \widehat{\wp}_2 \in \mathcal{P}$.

Example 2.5 Let us consider the IVPHFS $\widetilde{P}$ in Example 2.2. For the air pollution of the cities of Ankara and Istanbul, we have the following IVPHFEs:

$$\widehat{\wp}_1 = \{\{[0.2, 0.4]\}, \{[0.1, 0.3], [0.2, 0.4]\}, \{[0, 0.1], [0.1, 0.2]\}\}$$

and

$$\widehat{\wp}_2 = \{\{[0.1, 0.3]\}, \{[0.2, 0.2], [0.3, 0.4], [0.2, 0.5]\}, \{[0, 0.1]\}\}.$$

Then, we calculate as $\mathcal{S}(\widehat{\wp}_1) = 0.475 > 0.425 = \mathcal{S}(\widehat{\wp}_2)$, and so $\widehat{\wp}_1 \succ \widehat{\wp}_2$.

On the other hand, we consider the IVPHFE as

$$\widehat{\wp}_3 = \{\{[0.2, 0.4]\}, \{[0, 0.1], [0.1, 0.2]\}, \{[0.1, 0.3], [0.2, 0.4]\}\}.$$

Then, we obtain that $\mathcal{S}(\widehat{\wp}_1) = \mathcal{S}(\widehat{\wp}_2) = 0.475$ using score function and $\mathcal{A}(\widehat{\wp}_1) = \mathcal{A}(\widehat{\wp}_2) = 0.65$ using accuracy function. Using certainty function, we have $\mathcal{C}(\widehat{\wp}_1) = 0.6 > 0.525 = \mathcal{C}(\widehat{\wp}_2)$ and so $\widehat{\wp}_1 \succ \widehat{\wp}_2$.

Definition 2.6 Let $\widehat{\wp} = \{\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}\}$, $\widehat{\wp}_1 = \{\widehat{\alpha}_1, \widehat{\beta}_1, \widehat{\gamma}_1\}$ and $\widehat{\wp}_2 = \{\widehat{\alpha}_2, \widehat{\beta}_2, \widehat{\gamma}_2\}$ be three IVPH-FEs and ξ be any real number (i.e., $\xi > 0$). Then, the operations of IVPHFEs are described as

(a)

$$\widehat{\wp}_1 \cap \widehat{\wp}_2 = \bigcup_{\substack{[a_1^L, a_1^U] \in \widehat{\alpha}_1, [a_2^L, a_2^U] \in \widehat{\alpha}_2 \\ [b_1^L, b_1^U] \in \widehat{\beta}_1, [b_2^L, b_2^U] \in \widehat{\beta}_2 \\ [c_1^L, c_1^U] \in \widehat{\gamma}_1, [c_2^L, c_2^U] \in \widehat{\gamma}_2}} \left\{ \begin{array}{c} \{[a_1^L \wedge a_2^L, a_1^U \wedge a_2^U]\} \\ \{[b_1^L \vee b_2^L, b_1^U \vee b_2^U]\} \\ \{[c_1^L \vee c_2^L, c_1^U \vee c_2^U]\} \end{array} \right\}. \tag{2.6}$$

(b)

$$\widehat{\wp}_1 \cup \widehat{\wp}_2 = \bigcup_{\substack{[a_1^L, a_1^U] \in \widehat{\alpha}_1, [a_2^L, a_2^U] \in \widehat{\alpha}_2 \\ [b_1^L, b_1^U] \in \widehat{\beta}_1, [b_2^L, b_2^U] \in \widehat{\beta}_2 \\ [c_1^L, c_1^U] \in \widehat{\gamma}_1, [c_2^L, c_2^U] \in \widehat{\gamma}_2}} \left\{ \begin{array}{c} \{[a_1^L \vee a_2^L, a_1^U \vee a_2^U]\} \\ \{[b_1^L \wedge b_2^L, b_1^U \wedge b_2^U]\} \\ \{[c_1^L \wedge c_2^L, c_1^U \wedge c_2^U]\} \end{array} \right\}. \tag{2.7}$$

(c)

$$\widehat{\wp}^\circ = \bigcup_{[a^L, a^U] \in \widehat{\alpha}, [b^L, b^U] \in \widehat{\beta}, [c^L, c^U] \in \widehat{\gamma}} \left\{ \begin{array}{c} \{[c^L, c^U]\} \\ \{[b^L, b^U]\} \\ \{[a^L, a^U]\} \end{array} \right\}. \tag{2.8}$$

(d)

$$\widehat{\wp}_1 \oplus \widehat{\wp}_2 = \bigcup_{\substack{[a_1^L,a_1^U]\in\widehat{\alpha}_1,\ [a_2^L,a_2^U]\in\widehat{\alpha}_2 \\ [b_1^L,b_1^U]\in\widehat{\beta}_1,\ [b_2^L,b_2^U]\in\widehat{\beta}_2 \\ [c_1^L,c_1^U]\in\widehat{\gamma}_1,\ [c_2^L,c_2^U]\in\widehat{\gamma}_2}} \left\{ \begin{array}{l} \{[a_1^L + a_2^L - a_1^L a_2^L, a_1^U + a_2^U - a_1^U a_2^U]\} \\ \{[b_1^L b_2^L, b_1^U b_2^U]\} \\ \{[c_1^L c_2^L, c_1^U c_2^U]\} \end{array} \right\}.$$

(2.9)

(e)

$$\widehat{\wp}_1 \otimes \widehat{\wp}_2 = \bigcup_{\substack{[a_1^L,a_1^U]\in\widehat{\alpha}_1,[a_2^L,a_2^U]\in\widehat{\alpha}_2 \\ [b_1^L,b_1^U]\in\widehat{\beta}_1,[b_2^L,b_2^U]\in\widehat{\beta}_2 \\ [c_1^L,c_1^U]\in\widehat{\gamma}_1,[c_2^L,c_2^U]\in\widehat{\gamma}_2}} \left\{ \begin{array}{l} \{[a_1^L a_2^L, a_1^U a_2^U]\}, \\ \{[b_1^L + b_2^L - b_1^L b_2^L, b_1^U + b_2^U - b_1^U b_2^U]\} \\ \{[c_1^L + c_2^L - c_1^L c_2^L, c_1^U + c_2^U - c_1^U c_2^U]\} \end{array} \right\}.$$

(2.10)

(f)

$$\xi\widehat{\wp} = \bigcup_{[a^L,a^U]\in\widehat{\alpha},[b^L,b^U]\in\widehat{\beta},[c^L,c^U]\in\widehat{\gamma}} \left\{ \begin{array}{l} \{[1 - (1 - a^L)^n, 1 - (1 - a^U)^n]\} \\ \{[(b^L)^n, (b^U)^n]\} \\ \{[(c^L)^n, (c^U)^n]\} \end{array} \right\}. \quad (2.11)$$

(g)

$$\widehat{\wp}^{\xi} = \bigcup_{[a^L,a^U]\in\widehat{\alpha},[b^L,b^U]\in\widehat{\beta},[c^L,c^U]\in\widehat{\gamma}} \left\{ \begin{array}{l} \{[(a^L)^n, (a^U)^n]\} \\ \{[1 - (1 - b^L)^n, 1 - (1 - b^U)^n]\} \\ \{[1 - (1 - c^L)^n, 1 - (1 - c^U)^n]\} \end{array} \right\}. \quad (2.12)$$

Based on the aggregation principle and the operational laws of IVPHFEs, we describe some common interval-valued picture hesitant fuzzy aggregation operators.

Definition 2.7 Let $\widehat{\wp}_j$ $(j = 1, 2, ..., m)$ be a collection of IVPHFEs. In addition, let $\omega = (\omega_1, \omega_2, ..., \omega_m)^T$ be the weighted vector of $\widehat{\wp}_j$ $(j = 1, 2, ..., m)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{m} \omega_j = 1$.

(a) An interval-valued picture hesitant fuzzy weighted average (IVPHFWA) operator is a mapping $IVPHFWA_\omega : \mathcal{P}^m \to \mathcal{P}$, such that

$$IVPHFWA_\omega(\widehat{\wp}_1, \widehat{\wp}_2, ..., \widehat{\wp}_m) = \bigoplus_{j=1}^{m} \omega_j \widehat{\wp}_j = \omega_1 \widehat{\wp}_1 \oplus \omega_2 \widehat{\wp}_2 \oplus ... \oplus \omega_m \widehat{\wp}_m \quad (2.13)$$

(b) An interval-valued picture hesitant fuzzy weighted geometric (IVPHFWG) operator is a mapping $IVPHFWG_\omega : \mathcal{P}^m \to \mathcal{P}$, such that

$$IVPHFWG_\omega(\widehat{\wp}_1, \widehat{\wp}_2, ..., \widehat{\wp}_m) = \bigotimes_{j=1}^{m} \widehat{\wp}_j^{\omega_j} = \widehat{\wp}_1^{\omega_1} \otimes \widehat{\wp}_2^{\omega_2} \otimes ... \otimes \widehat{\wp}_m^{\omega_m} \quad (2.14)$$

Theorem 2.8 Let $\widehat{\wp}_j$ $(j = 1, 2, ..., m)$ be a collection of IVPHFEs. In addition, let $\omega = (\omega_1, \omega_2, ..., \omega_m)^T$ be the weighted vector of $\widehat{\wp}_j$ $(j = 1, 2, ..., m)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{m} \omega_j = 1$.

1. *The aggregation value of IVPHFEs using the IVPHFWA operator is also an IVPHFE, and*

$$IVPHFWA_\omega(\widehat{\wp}_1, \widehat{\wp}_2, ..., \widehat{\wp}_m)$$

$$= \bigoplus_{j=1}^m \omega_j \widehat{\wp}_j = \bigcup_{\substack{[a_j^L, a_j^U] \in \widehat{\alpha}_j \ (j=1,2,...,m) \\ [b_j^L, b_j^U] \in \widehat{\beta}_j \ (j=1,2,...,m) \\ [c_j^L, c_j^U] \in \widehat{\gamma}_j \ (j=1,2,...,m)}} \left\{ \left\{ \left[1 - \prod_{j=1}^m (1-a_j^L)^{\omega_j}, 1 - \prod_{j=1}^m (1-a_j^U)^{\omega_j} \right] \right\}, \left[\prod_{j=1}^m (b_j^L)^{\omega_j}, \prod_{j=1}^m (b_j^U)^{\omega_j} \right], \left[\prod_{j=1}^m (c_j^L)^{\omega_j}, \prod_{j=1}^m (c_j^U)^{\omega_j} \right] \right\}.$$

(2.15)

2. *The aggregation value of IVPHFEs using the IVPHFWG operator is also an IVPHFE, and*

$$IVPHFWG_\omega(\widehat{\wp}_1, \widehat{\wp}_2, ..., \widehat{\wp}_m)$$

$$= \bigotimes_{j=1}^m \widehat{\wp}_j^{\omega_j} = \bigcup_{\substack{[a_j^L, a_j^U] \in \widehat{\alpha}_j \ (j=1,2,...,m) \\ [b_j^L, b_j^U] \in \widehat{\beta}_j \ (j=1,2,...,m) \\ [c_j^L, c_j^U] \in \widehat{\gamma}_j \ (j=1,2,...,m)}} \left\{ \left\{ \left[\prod_{j=1}^m (a_j^L)^{\omega_j}, \prod_{j=1}^m (a_j^U)^{\omega_j} \right] \right\}, \left[1 - \prod_{j=1}^m (1-b_j^L)^{\omega_j}, 1 - \prod_{j=1}^m (1-b_j^U)^{\omega_j} \right], \left[1 - \prod_{j=1}^m (1-c_j^L)^{\omega_j}, 1 - \prod_{j=1}^m (1-c_j^U)^{\omega_j} \right] \right\}.$$

(2.16)

Proof The proofs are similar to the proof of Theorem 3.6, so the repetition is avoided. □

Example 2.9 We consider $\widehat{\wp}_1$ and $\widehat{\wp}_2$ given in Example 2.5. From Example 2.2, we have the following IVPHFEs for the air pollution of the cities of Yozgat and Antalya, respectively:

$$\widehat{\wp}_3 = \{\{[0.2, 0.35], [0.3, 0.4]\}, \{[0.1, 0.25], [0.2, 0.3]\}, \{[0.1, 0.2]\}\}$$

and

$$\widehat{\wp}_4 = \{\{[0.1, 0.3], [0.2, 0.5]\}, \{[0.2, 0.2]\}, \{[0.1, 0.1]\}\}.$$

Then, we have

$$\widehat{\wp}_1 \cap \widehat{\wp}_4 = \{\{[0.1, 0.3], [0.2, 0.4]\}, \{[0.2, 0.3], [0.2, 0.4]\}, \{[0.1, 0.1], [0.1, 0.2]\}\},$$
$$\widehat{\wp}_1 \cup \widehat{\wp}_4 = \{\{[0.2, 0.4], [0.2, 0.5]\}, \{[0.1, 0.2], [0.2, 0.2]\}, \{[0, 0.1], [0.1, 0.1]\}\},$$
$$\widehat{\wp}_3^\circ = \{\{[0.1, 0.2]\}, \{[0.1, 0.25], [0.2, 0.3]\}, \{[0.2, 0.35], [0.3, 0.4]\}\},$$

Assume that $\omega = (\omega_1, \omega_2, \omega_3) = (0.2, 0.3, 0.5)^T$ for $\widehat{\wp}_1$, $\widehat{\wp}_2$ and $\widehat{\wp}_3$. Then, we obtain

$$IVPHFWA_\omega(\widehat{\wp}_1, \widehat{\wp}_2, \widehat{\wp}_3) = \bigoplus_{j=1}^3 \omega_j \widehat{\wp}_j$$

$$= \left\{ \left\{ \begin{array}{l} \{[0.1712, 0.3459], [0.2247, 0.3716]\}, \\ [0.1213, 0.2424], [0.1741, 0.3067], [0.139, 0.2985], [0.1966, 0.3776], \\ [0.1231, 0.3192], [0.1741, 0.4037], [0.1414, 0.2568], [0.2, 0.3249], \\ [0.1597, 0.3162], [0.2258, 0.4], [0.1414, 0.3381], [0.2, 0.4276] \\ \{[0, 0.1414], [0, 0.1624]\} \end{array} \right\} \right\},$$

and

$$IVPHFWG_\omega(\widehat{\wp}_1, \widehat{\wp}_2, \widehat{\wp}_3) = \bigotimes_{j=1}^{3} \widehat{\wp}_j^{\omega_j}$$

$$= \left\{ \begin{array}{l} \left\{ [0.1624, 0.3432], [0.1989, 0.3669] \right\}, \\ \left\{ \begin{array}{l} [0.1312, 0.2458], [0.1809, 0.3254], [0.1653, 0.3081], [0.213, 0.3812], \\ [0.1312, 0.345], [0.1809, 0.4141], [0.1514, 0.2687], [0.2, 0.3459], \\ [0.1847, 0.3291], [0.2314, 0.4], [0.1514, 0.3648], [0.2, 0.4319] \end{array} \right\} \\ \left\{ [0.0513, 0.1514], [0.071, 0.1712] \right\} \end{array} \right\}.$$

All the operations, relations, and operators of IVPHFEs can only be used to deal with time-independent IVPHF arguments. However, these notions are not satisfactory if the IVPHF information may be collected at the different periods, where the period duration is a point or portion of the time.

3 Dynamic interval-valued picture hesitant fuzzy weighted aggregation operators

In this part, we define some dynamic interval-valued picture hesitant fuzzy weighted aggregation operators to aggregate interval-valued picture hesitant fuzzy information collected at the different periods, where the period duration is a point or portion of the time.

Definition 3.1 Let t be the variable of time, then we define an interval-valued picture hesitant fuzzy variable (IVPHFV) as $\widehat{\wp}(t) = \{\widehat{\alpha}(t), \widehat{\beta}(t), \widehat{\gamma}(t)\} = \bigcup_{[a_{(t)}^L, a_{(t)}^U] \in \widehat{\alpha}_{(t)}, [b_{(t)}^L, b_{(t)}^U] \in \widehat{\beta}_{(t)}, [c_{(t)}^L, c_{(t)}^U] \in \widehat{\gamma}_{(t)}}$

$\{\{[a_{(t)}^L, a_{(t)}^U]\}, \{[b_{(t)}^L, b_{(t)}^U]\}, \{[c_{(t)}^L, c_{(t)}^U]\}\}$.

For an IVPHFV $\widehat{\wp}(t) = \{\widehat{\alpha}(t), \widehat{\beta}(t), \widehat{\gamma}(t)\}$, if $t = t_1, t_2, ..., t_k$ then $\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_k)$ is represented as k interval-valued picture hesitant fuzzy elements collected at k periods. Now, we introduce some operations related to the values of IVPHFVs.

Definition 3.2 Let $\widehat{\wp}(t_1) = \{\widehat{\alpha}(t_1), \widehat{\beta}(t_1), \widehat{\gamma}(t_1)\}$ and $\widehat{\wp}(t_2) = \{\widehat{\alpha}(t_2), \widehat{\beta}(t_2), \widehat{\gamma}(t_2)\}$ be two IVPHFVs and ξ be any positive real number (i.e., $\xi > 0$). Then, the operations of IVPHFVs are described as

(a)

$\widehat{\wp}(t_1) \oplus \widehat{\wp}(t_2)$

$$= \bigcup_{\substack{[a_{(t_1)}^L, a_{(t_1)}^U] \in \widehat{\alpha}(t_1), [a_{(t_2)}^L, a_{(t_2)}^U] \in \widehat{\alpha}(t_2) \\ [b_{(t_1)}^L, b_{(t_1)}^U] \in \widehat{\beta}(t_1), [b_{(t_2)}^L, b_{(t_2)}^U] \in \widehat{\beta}(t_2) \\ [c_{(t_1)}^L, c_{(t_1)}^U] \in \widehat{\gamma}(t_1), [c_{(t_2)}^L, c_{(t_2)}^U] \in \widehat{\gamma}(t_1)}} \left\{ \begin{array}{l} \left\{ \left[a_{(t_1)}^L + a_{(t_2)}^L - a_{(t_1)}^L a_{(t_2)}^L, a_{(t_1)}^U + a_{(t_2)}^U - a_{(t_1)}^U a_{(t_2)}^U \right] \right\} \\ \left\{ \left[b_{(t_1)}^L b_{(t_2)}^L, b_{(t_1)}^U b_{(t_2)}^U \right] \right\} \\ \left\{ \left[c_{t_1}^L c_{(t_2)}^L, c_{(t_1)}^U c_{(t_2)}^U \right] \right\} \end{array} \right\}.$$

(3.1)

(b)

$\widehat{\wp}(t_1) \otimes \widehat{\wp}(t_2)$

$$= \bigcup_{\substack{[a^L_{(t_1)},a^U_{(t_1)}]\in\widehat{\alpha}(t_1),\ [a^L_{(t_2)},a^U_{(t_2)}]\in\widehat{\alpha}(t_2) \\ [b^L_{(t_1)},b^U_{(t_1)}]\in\widehat{\beta}(t_1),\ [b^L_{(t_2)},b^U_{(t_2)}]\in\widehat{\beta}(t_2) \\ [c^L_{(t_1)},c^U_{(t_1)}]\in\widehat{\gamma}(t_1),\ [c^L_{(t_2)},c^U_{(t_2)}]\in\widehat{\gamma}(t_2)}} \left\{ \begin{array}{l} \left\{\left[a^L_{(t_1)}a^L_{(t_2)},\ a^U_{(t_1)}a^U_{(t_2)}\right]\right\} \\ \left\{\left[b^L_{(t_1)}+b^L_{(t_2)}-b^L_{(t_1)}b^L_{(t_2)},\ b^U_{(t_1)}+b^U_{(t_2)}-b^U_{(t_1)}b^U_{(t_2)}\right]\right\} \\ \left\{\left[c^L_{(t_1)}+c^L_{(t_2)}-c^L_{(t_1)}c^L_{(t_2)},\ c^U_{(t_1)}+c^U_{(t_2)}-c^U_{(t_1)}c^U_{(t_2)}\right]\right\} \end{array} \right\}.$$

(3.2)

(c)

$$\xi\widehat{\wp}(t_1) = \bigcup_{\substack{[a^L_{(t_1)},a^U_{(t_1)}]\in\widehat{\alpha}(t_1) \\ [b^L_{(t_1)},b^U_{(t_1)}]\in\widehat{\beta}(t_1) \\ [c^L_{(t_1)},c^U_{(t_1)}]\in\widehat{\gamma}(t_1)}} \left\{ \begin{array}{l} \left\{\left[1-(1-a^L_{(t_1)})^n,\ 1-(1-a^U_{(t_1)})^n\right]\right\} \\ \left\{\left[(b^L_{(t_1)})^n,\ (b^U_{(t_1)})^n\right]\right\} \\ \left\{\left[(c^L_{(t_1)})^n,\ (c^U_{(t_1)})^n\right]\right\} \end{array} \right\}.$$

(3.3)

(d)

$$(\widehat{\wp}(t_1))^{\xi} = \bigcup_{\substack{[a^L_{(t_1)},a^U_{(t_1)}]\in\widehat{\alpha}(t_1) \\ [b^L_{(t_1)},b^U_{(t_1)}]\in\widehat{\beta}(t_1) \\ [c^L_{(t_1)},c^U_{(t_1)}]\in\widehat{\gamma}(t_1)}} \left\{ \begin{array}{l} \left\{\left[(a^L_{(t_1)})^n,\ (a^U_{(t_1)})^n\right]\right\} \\ \left\{\left[1-(1-b^L_{(t_1)})^n,\ 1-(1-b^U_{(t_1)})^n\right]\right\} \\ \left\{\left[1-(1-c^L_{(t_1)})^n,\ 1-(1-c^U_{(t_1)})^n\right]\right\} \end{array} \right\}.$$

(3.4)

Example 3.3 We consider Example 2.2. For the city of Yozgat, let us assume that the air pollution is evaluated according to the interval-valued picture hesitant fuzzy information collected in two different periods. According to the two-period evaluations, we consider two following IVPHFEs:

For the period t_1,

$$\widehat{\wp}(t_1) = \{\{[0.2, 0.35], [0.3, 0.4]\}, \{[0.1, 0.25], [0.2, 0.3]\}, \{[0.1, 0.2]\}\}$$

and for the period t_2,

$$\widehat{\wp}(t_2) = \{\{[0.1, 0.15]\}, \{[0.1, 0.2], [0.15, 0.3], [0.2, 0.4]\}, \{[0, 0.1]\}\}.$$

In addition, let $\xi = 4$. Then, we obtain

$\widehat{\wp}(t_1) \oplus \widehat{\wp}(t_2) = \{\{[0.28, 0.4475], [0.37, 0.49]\}, \{[0.01, 0.05], [0.015, 0.075], [0.02, 0.1],$
  $[0.02, 0.06], [0.03, 0.09], [0.04, 0.12]\}, \{[0, 0.02]\}\}$,
$\widehat{\wp}(t_1) \otimes \widehat{\wp}(t_2) = \{\{[0.02, 0.0525], [0.03, 0.06]\}, \{[0.19, 0.4], [0.235, 0.475], [0.28, 0.55],$
  $[0.28, 0.44], [0.32, 0.51], [0.36, 0.58]\}, \{[0.1, 0.28]\}\}$,
$4\widehat{\wp}(t_1) = \{\{[0.5904, 0.8215], [0.76, 0.8704]\}, \{[0.0001, 0.0039], [0.0016, 0.0081]\}, \{[0.0001, 0.0016]\}\}$,
$\widehat{\wp}(t_2)^4 = \{\{[0.0001, 0.0005]\}, \{[0.3439, 0.4096], [0.478, 0.76], [0.4096, 0.8704]\}, \{[0, 0.3439]\}\}$.

Proposition 3.4 *Let $\widehat{\wp}(t_1) = \{\widehat{\alpha}(t_1), \widehat{\beta}(t_1), \widehat{\gamma}(t_1)\}$ and $\widehat{\wp}(t_2) = \{\widehat{\alpha}(t_2), \widehat{\beta}(t_2), \widehat{\gamma}(t_2)\}$ be two IVPHFVs and $\xi > 0$.*

1. $\widehat{\wp}(t_1) \oplus \widehat{\wp}(t_2) = \widehat{\wp}(t_2) \oplus \widehat{\wp}(t_1)$.
2. $\widehat{\wp}(t_1) \otimes \widehat{\wp}(t_2) = \widehat{\wp}(t_2) \otimes \widehat{\wp}(t_1)$.
3. $\xi(\widehat{\wp}(t_1) \oplus \widehat{\wp}(t_2)) = \xi\widehat{\wp}(t_1) \oplus \xi\widehat{\wp}(t_2)$.
4. $(\widehat{\wp}(t_1) \otimes \widehat{\wp}(t_2))^{\xi} = (\widehat{\wp}(t_1))^{\xi} \otimes (\widehat{\wp}(t_2))^{\xi}$.

Proof It is trivial from Definition 3.2.   □

By combining the above operations described for IVPHFVs, the following weighted aggregation operators are developed.

Definition 3.5 Let $\widehat{\wp}(t_k)$ $(k = 1, 2, ..., r)$ be the IVPHFEs collected from k different periods t_k $(k = 1, 2, ..., r)$. In addition, let $\omega(t) = (\omega(t_1), \omega(t_2), ..., \omega(t_r))^T$ be the weighted vector of t_k $(k = 1, 2, ..., r)$, with $\omega(t_k) \in [0, 1]$ and $\sum_{k=1}^{r} \omega(t_k) = 1$, and $\lambda > 0$.

(a) A dynamic interval-valued picture hesitant fuzzy weighted average (DIVPHFWA) operator is a mapping $DIVPHFWA_{\omega(t)} : \mathcal{P}^r \to \mathcal{P}$, such that

$$DIVPHFWA_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) = \bigoplus_{k=1}^{r} \omega(t_k)\widehat{\wp}(t_k)$$
$$= \omega(t_1)\widehat{\wp}(t_1) \oplus \omega(t_2)\widehat{\wp}(t_2) \oplus ... \oplus \omega(t_r)\widehat{\wp}(t_r) \tag{3.5}$$

(b) A dynamic interval-valued picture hesitant fuzzy weighted geometric (DIVPHFWG) operator is a mapping $DIVPHFWG_{\omega(t)} : \mathcal{P}^r \to \mathcal{P}$, such that

$$DIVPHFWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) = \bigotimes_{k=1}^{r} (\widehat{\wp}(t_k))^{\omega(t_k)}$$
$$= (\widehat{\wp}(t_1))^{\omega(t_1)} \otimes (\widehat{\wp}(t_2))^{\omega(t_2)} \otimes ... \otimes (\widehat{\wp}(t_k))^{\omega(t_k)} \tag{3.6}$$

(c) A generalized dynamic interval-valued picture hesitant fuzzy weighted average (GDIVPHFWA) operator is a mapping $GDIVPHFWA_{\omega(t)} : \mathcal{P}^r \to \mathcal{P}$ such that

$$GDIVPHFWA_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) = \left(\bigoplus_{k=1}^{r} \omega(t_k)(\widehat{\wp}(t_k))^{\lambda} \right)^{\frac{1}{\lambda}}$$
$$= (\omega(t_1)(\widehat{\wp}(t_1))^{\lambda} \oplus \omega(t_2)(\widehat{\wp}(t_2))^{\lambda} \oplus ... \oplus \omega(t_r)(\widehat{\wp}(t_r))^{\lambda})^{\frac{1}{\lambda}} \tag{3.7}$$

(d) A generalized dynamic interval-valued picture hesitant fuzzy weighted geometric (GDIVPHFWG) operator is a mapping $GDIVPHFWG_{\omega(t)} : \mathcal{P}^r \to \mathcal{P}$ such that

$$GDIVPHFWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) = \frac{1}{\lambda} \left(\bigotimes_{k=1}^{r} \lambda(\widehat{\wp}(t_k))^{\omega(t_k)} \right)$$
$$= \frac{1}{\lambda} \left(\lambda(\widehat{\wp}(t_1))^{\omega(t_1)} \otimes \lambda(\widehat{\wp}(t_2))^{\omega(t_2)} \otimes ... \otimes \lambda(\widehat{\wp}(t_r))^{\omega(t_r)} \right) \tag{3.8}$$

Theorem 3.6 *Let $\widehat{\wp}(t_k)$ $(k = 1, 2, ..., r)$ be the IVPHFEs collected from k different periods t_k $(k = 1, 2, ..., r)$. In addition, let $\omega(t) = (\omega(t_1), \omega(t_2), ..., \omega(t_r))^T$ be the weighted vector of t_k $(k = 1, 2, ..., r)$, with $\omega(t_k) \in [0, 1]$ and $\sum_{k=1}^{r} \omega(t_k) = 1$, and $\lambda > 0$.*

1. *The aggregation value of IVPHFEs using the DIVPHFWA operator is also an IVPHFE, and*

$$DIVPHFWA_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r))$$

$$= \bigcup_{\substack{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k) \ (k=1,2,...,r) \\ [b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k) \ (k=1,2,...,r) \\ [c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k) \ (k=1,2,...,r)}} \left\{ \begin{array}{l} \left\{ \left[1 - \prod_{k=1}^{r} (1 - a_{(t_k)}^L)^{\omega(t_k)}, 1 - \prod_{k=1}^{r} (1 - a_{(t_k)}^U)^{\omega(t_k)} \right] \right\} \\ \left\{ \left[\prod_{k=1}^{r} (b_{(t_k)}^L)^{\omega(t_k)}, \prod_{k=1}^{r} (b_{(t_k)}^U)^{\omega(t_k)} \right] \right\} \\ \left\{ \left[\prod_{k=1}^{r} (c_{(t_k)}^L)^{\omega(t_k)}, \prod_{k=1}^{r} (c_{(t_k)}^U)^{\omega(t_k)} \right] \right\} \end{array} \right\}. \tag{3.9}$$

2. *The aggregation value of IVPHFEs using the DIVPHFWG operator is also an IVPHFE,*
   and

$$DIVPHFWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r))$$

$$= \bigcup_{\substack{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k) \ (k=1,2,...,r) \\ [b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k) \ (k=1,2,...,r) \\ [c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k) \ (k=1,2,...,r)}} \left\{ \begin{array}{l} \left\{ \left[\prod_{k=1}^{r} (a_{(t_k)}^L)^{\omega(t_k)}, \prod_{k=1}^{r} (a_{(t_k)}^U)^{\omega(t_k)} \right] \right\} \\ \left\{ \left[1 - \prod_{k=1}^{r} (1 - b_{(t_k)}^L)^{\omega(t_k)}, 1 - \prod_{k=1}^{r} (1 - b_{(t_k)}^U)^{\omega(t_k)} \right] \right\} \\ \left\{ \left[1 - \prod_{k=1}^{r} (1 - c_{(t_k)}^L)^{\omega(t_k)}, 1 - \prod_{k=1}^{r} (1 - c_{(t_k)}^U)^{\omega(t_k)} \right] \right\} \end{array} \right\}.$$

$$(3.10)$$

3. *The aggregation value of IVPHFEs using the GDIVPHFWA operator is also an IVPHFE,*
   and

$$GDIVPHFWA_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r))$$

$$= \bigcup_{\substack{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k) \ (k=1,2,...,r) \\ [b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k) \ (k=1,2,...,r) \\ [c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k) \ (k=1,2,...,r)}} \left\{ \begin{array}{l} \left\{ \left[\left(1 - \prod_{k=1}^{r} (1 - (a_{(t_k)}^L)^{\lambda})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{k=1}^{r} (1 - (a_{(t_k)}^U)^{\lambda})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \right] \right\} \\ \left\{ \left[1 - \left(1 - \prod_{k=1}^{r} (1 - (1 - b_{(t_k)}^L)^{\lambda})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{k=1}^{r} (1 - (1 - b_{(t_k)}^U)^{\lambda})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \right] \right\} \\ \left\{ \left[1 - \left(1 - \prod_{k=1}^{r} (1 - (1 - c_{(t_k)}^L)^{\lambda})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{k=1}^{r} (1 - (1 - c_{(t_k)}^U)^{\lambda})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \right] \right\} \end{array} \right\}.$$

$$(3.11)$$

(4) The aggregation value of IVPHFEs using the GDIVPHFWG operator is also an IVPHFE,
   and

$$GDIVPHFWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r))$$

$$= \bigcup_{\substack{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k) \ (k=1,2,...,r) \\ [b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k) \ (k=1,2,...,r) \\ [c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k) \ (k=1,2,...,r)}} \left\{ \begin{array}{l} \left\{ \left[1 - \left(1 - \prod_{k=1}^{r} \left(1 - (1 - a_{(t_k)}^L)^{\lambda}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{k=1}^{r} (1 - (1 - a_{(t_k)}^U)^{\lambda})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \right] \right\} \\ \left\{ \left[\left(1 - \prod_{k=1}^{r} (1 - (b_{(t_k)}^L)^{\lambda})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{k=1}^{r} (1 - (b_{(t_k)}^U)^{\lambda})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \right] \right\} \\ \left\{ \left[\left(1 - \prod_{k=1}^{r} (1 - (c_{(t_k)}^L)^{\lambda})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{k=1}^{r} (1 - (c_{(t_k)}^U)^{\lambda})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \right] \right\} \end{array} \right\}.$$

$$(3.12)$$

Proof Let us prove (4), the others can be demonstrate similar to this.

We use the method of mathematical induction for proof.

- For $r = 1$:
  By Definition 3.2 (d), we calculate

$$\lambda \widehat{\wp}(t_1) = \bigcup_{\substack{[a_{(t_1)}^L, a_{(t_1)}^U] \in \widehat{\alpha}_{(t_1)} \\ [b_{(t_1)}^L, b_{(t_1)}^U] \in \widehat{\beta}(t_1) \\ [c_{(t_1)}^L, c_{(t_1)}^U] \in \widehat{\gamma}(t_1)}} \left\{ \begin{array}{l} \left\{ \left[1 - (1 - a_{(t_1)}^L)^{\lambda}, 1 - (1 - a_{(t_1)}^U)^{\lambda} \right] \right\} \\ \left\{ \left[(b_{(t_1)}^L)^{\lambda}, (b_{(t_1)}^U)^{\lambda} \right] \right\} \\ \left\{ \left[(c_{(t_1)}^L)^{\lambda}, (c_{(t_1)}^U)^{\lambda} \right] \right\} \end{array} \right\}$$

and moreover

$$\frac{1}{\lambda}(\lambda\widehat{\wp}(t_1)) = \bigcup_{\substack{[a^L_{(t_1)},a^U_{(t_1)}]\in\widehat{\alpha}_{(t_1)} \\ [b^L_{(t_1)},b^U_{(t_1)}]\in\widehat{\beta}(t_1) \\ [c^L_{(t_1)},c^U_{(t_1)}]\in\widehat{\gamma}(t_1)}} \left\{ \begin{array}{l} \left\{\left[1-(1-(1-(1-a^L_{(t_1)})^\lambda))^{\frac{1}{\lambda}}, 1-(1-(1-(1-a^U_{(t_1)})^\lambda))^{\frac{1}{\lambda}}\right]\right\} \\ \left\{\left[((b^L_{(t_1)})^\lambda)^{\frac{1}{\lambda}}, ((b^U_{(t_1)})^\lambda)^{\frac{1}{\lambda}}\right]\right\} \\ \left\{\left[((c^L_{(t_1)})^\lambda)^{\frac{1}{\lambda}}, ((c^U_{(t_1)})^\lambda)^{\frac{1}{\lambda}}\right]\right\} \end{array} \right\}$$

$$= \bigcup_{\substack{[a^L_{(t_1)},a^U_{(t_1)}]\in\widehat{\alpha}_{(t_1)} \\ [b^L_{(t_1)},b^U_{(t_1)}]\in\widehat{\beta}(t_1) \\ [c^L_{(t_1)},c^U_{(t_1)}]\in\widehat{\gamma}(t_1)}} \left\{ \begin{array}{l} \left\{\left[a^L_{(t_1)}, a^U_{(t_1)}\right]\right\} \\ \left\{\left[b^L_{(t_1)}, b^U_{(t_1)}\right]\right\} \\ \left\{\left[c^L_{(t_1)}, c^U_{(t_1)}\right]\right\} \end{array} \right\}.$$

Hence, we have $GDIVPHFWG_{\omega(t)}(\widehat{\wp}(t_1)) = \frac{1}{\lambda}((\lambda\widehat{\wp}(t_1))^{\omega(t_1)}) = \widehat{\wp}(t_1)$ and so Eq. (3.12) is valid for $r = 1$.

- For $r = 2$:

  Since (from Definition 2.6 (c) and (d))

$$\lambda(\widehat{\wp}(t_1))^{\omega(t_1)} = \bigcup_{\substack{[a^L_{(t_1)},a^U_{(t_1)}]\in\widehat{\alpha}(t_1) \\ [b^L_{(t_1)},b^U_{(t_1)}]\in\widehat{\beta}(t_1) \\ [c^L_{(t_1)},c^U_{(t_1)}]\in\widehat{\gamma}(t_1)}} \left\{ \begin{array}{l} \left\{\left[(1-(1-a^L_{(t_1)})^\lambda)^{\omega(t_1)}, (1-(1-a^U_{(t_1)})^\lambda)^{\omega(t_1)}\right]\right\} \\ \left\{\left[1-(1-(b^L_{(t_1)})^\lambda)^{\omega(t_1)}, 1-(1-(b^U_{(t_1)})^\lambda)^{\omega(t_1)}\right]\right\} \\ \left\{\left[1-(1-(c^L_{(t_1)})^\lambda)^{\omega(t_1)}, 1-(1-(c^U_{(t_1)})^\lambda)^{\omega(t_1)}\right]\right\} \end{array} \right\}$$

and

$$\lambda(\widehat{\wp}(t_2))^{\omega(t_2)} = \bigcup_{\substack{[a^L_{(t_2)},a^U_{(t_2)}]\in\widehat{\alpha}(t_2) \\ [b^L_{(t_2)},b^U_{(t_2)}]\in\widehat{\beta}(t_2) \\ [c^L_{(t_2)},c^U_{(t_2)}]\in\widehat{\gamma}(t_2)}} \left\{ \begin{array}{l} \left\{\left[(1-(1-a^L_{(t_2)})^\lambda)^{\omega(t_2)}, (1-(1-a^U_{(t_2)})^\lambda)^{\omega(t_2)}\right]\right\} \\ \left\{\left[1-(1-(b^L_{(t_2)})^\lambda)^{\omega(t_2)}, 1-(1-(b^U_{(t_2)})^\lambda)^{\omega(t_2)}\right]\right\} \\ \left\{\left[1-(1-(c^L_{(t_2)})^\lambda)^{\omega(t_2)}, 1-(1-(c^U_{(t_2)})^\lambda)^{\omega(t_2)}\right]\right\} \end{array} \right\},$$

we found that (from Definition 2.6 (b))

$$\lambda(\widehat{\wp}(t_1))^{\omega(t_1)} \otimes \lambda(\widehat{\wp}(t_2))^{\omega(t_2)}$$

$$= \bigcup_{\substack{[a^L_{(t_k)},a^U_{(t_k)}]\in\widehat{\alpha}(t_k)\ (k=1,2) \\ [b^L_{(t_k)},b^U_{(t_k)}]\in\widehat{\beta}(t_k)\ (k=1,2) \\ [c^L_{(t_k)},c^U_{(t_k)}]\in\widehat{\gamma}(t_k)\ (k=1,2)}} \left\{ \begin{array}{l} \left\{\left[(1-(1-a^L_{(t_1)})^\lambda)^{\omega(t_1)}(1-(1-a^L_{(t_2)})^\lambda)^{\omega(t_2)}, (1-(1-a^U_{(t_1)})^\lambda)^{\omega(t_1)}(1-(1-a^U_{(t_2)})^\lambda)^{\omega(t_2)}\right]\right\} \\ \left\{\begin{array}{l}\left[1-(1-(b^L_{(t_1)})^\lambda)^{\omega(t_1)}+1-(1-(b^L_{(t_2)})^\lambda)^{\omega(t_2)}-(1-(1-(b^L_{(t_1)})^\lambda)^{\omega(t_1)})(1-(1-(b^L_{(t_2)})^\lambda)^{\omega(t_2)})\right. \\ \left.1-(1-(b^U_{(t_1)})^\lambda)^{\omega(t_1)}+1-(1-(b^U_{(t_2)})^\lambda)^{\omega(t_2)}-(1-(1-(b^U_{(t_1)})^\lambda)^{\omega(t_1)})(1-(1-(b^U_{(t_2)})^\lambda)^{\omega(t_2)})\right]\end{array}\right\} \\ \left\{\begin{array}{l}\left[1-(1-(c^L_{(t_1)})^\lambda)^{\omega(t_1)}+1-(1-(c^L_{(t_2)})^\lambda)^{\omega(t_2)}-(1-(1-(c^L_{(t_1)})^\lambda)^{\omega(t_1)})(1-(1-(c^L_{(t_2)})^\lambda)^{\omega(t_2)})\right. \\ \left.1-(1-(c^U_{(t_1)})^\lambda)^{\omega(t_1)}+1-(1-(c^U_{(t_2)})^\lambda)^{\omega(t_2)}-(1-(1-(c^U_{(t_1)})^\lambda)^{\omega(t_1)})(1-(1-(c^U_{(t_2)})^\lambda)^{\omega(t_2)})\right]\end{array}\right\} \end{array} \right\}$$

$$= \bigcup_{\substack{[a^L_{(t_k)},a^U_{(t_k)}]\in\widehat{\alpha}(t_k)\ (k=1,2) \\ [b^L_{(t_k)},b^U_{(t_k)}]\in\widehat{\beta}(t_k)\ (k=1,2) \\ [c^L_{(t_k)},c^U_{(t_k)}]\in\widehat{\gamma}(t_k)\ (k=1,2)}} \left\{ \begin{array}{l} \left\{\left[(1-(1-a^L_{(t_1)})^\lambda)^{\omega(t_1)}(1-(1-a^L_{(t_2)})^\lambda)^{\omega(t_2)}, (1-(1-a^U_{(t_1)})^\lambda)^{\omega(t_1)}(1-(1-a^U_{(t_2)})^\lambda)^{\omega(t_2)}\right]\right\} \\ \left\{\left[1-(1-(b^L_{(t_1)})^\lambda)^{\omega(t_1)}(1-(b^L_{(t_2)})^\lambda)^{\omega(t_2)}, 1-(1-(b^U_{(t_1)})^\lambda)^{\omega(t_1)}(1-(b^U_{(t_2)})^\lambda)^{\omega(t_2)}\right]\right\} \\ \left\{\left[1-(1-(c^L_{(t_1)})^\lambda)^{\omega(t_1)}(1-(c^L_{(t_2)})^\lambda)^{\omega(t_2)}, 1-(1-(c^U_{(t_1)})^\lambda)^{\omega(t_1)}(1-(c^U_{(t_2)})^\lambda)^{\omega(t_2)}\right]\right\} \end{array} \right\}$$

$$
= \bigcup_{\substack{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k) \ (k=1,2) \\ [b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k) \ (k=1,2) \\ [c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k) \ (k=1,2)}} \left\{ \begin{array}{l} \left\{ \left[\prod_{k=1}^{2}(1-(1-a_{(t_k)}^L)^\lambda)^{\omega(t_k)}, \prod_{k=1}^{2}(1-(1-a_{(t_k)}^U)^\lambda)^{\omega(t_k)} \right] \right\} \\ \left\{ \left[\prod_{k=1}^{2}(1-(b_{(t_k)}^L)^\lambda)^{\omega(t_k)}, \prod_{k=1}^{2}(1-(b_{(t_k)}^U)^\lambda)^{\omega(t_k)} \right] \right\} \\ \left\{ \left[\prod_{k=1}^{2}(1-(c_{(t_k)}^L)^\lambda)^{\omega(t_k)}, \prod_{k=1}^{2}(1-(c_{(t_k)}^U)^\lambda)^{\omega(t_k)} \right] \right\} \end{array} \right\} \tag{3.13}
$$

and thus

$$
\frac{1}{\lambda}\left(\lambda(\widehat{\wp}(t_1))^{\omega(t_1)} \otimes \lambda(\widehat{\wp}(t_1))^{\omega(t_1)}\right)
$$

$$
= \bigcup_{\substack{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k) \ (k=1,2) \\ [b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k) \ (k=1,2) \\ [c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k) \ (k=1,2)}} \left\{ \begin{array}{l} \left\{ \left[1-\left(1-\prod_{k=1}^{2}(1-(1-a_{(t_k)}^L)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, 1-\left(1-\prod_{k=1}^{2}(1-(1-a_{(t_k)}^U)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \right] \right\} \\ \left\{ \left[\left(1-\prod_{k=1}^{2}(1-(b_{(t_k)}^L)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, \left(1-\prod_{k=1}^{2}(1-(b_{(t_k)}^U)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \right] \right\} \\ \left\{ \left[\left(1-\prod_{k=1}^{2}(1-(c_{(t_k)}^L)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, \left(1-\prod_{k=1}^{2}(1-(c_{(t_k)}^U)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \right] \right\} \end{array} \right\}
$$

$$
= GDIVPHFWG_\epsilon^{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2)).
$$

Hence, Eq. (3.12) holds for $r = 2$.

- For $r = s$:

If Eq. (3.12) is valid, then we obtain the following

$$
GDIVPHFWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_s))
$$

$$
= \bigcup_{\substack{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k) \ (k=1,2,...,s) \\ [b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k) \ (k=1,2,...,s) \\ [c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k) \ (k=1,2,...,s)}} \left\{ \begin{array}{l} \left\{ \left[1-\left(1-\prod_{k=1}^{s}(1-(1-a_{(t_k)}^L)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, 1-\left(1-\prod_{k=1}^{s}(1-(1-a_{(t_k)}^U)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \right] \right\} \\ \left\{ \left[\left(1-\prod_{k=1}^{s}(1-(b_{(t_k)}^L)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, \left(1-\prod_{k=1}^{s}(1-(b_{(t_k)}^U)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \right] \right\} \\ \left\{ \left[\left(1-\prod_{k=1}^{s}(1-(c_{(t_k)}^L)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, \left(1-\prod_{k=1}^{s}(1-(c_{(t_k)}^U)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \right] \right\} \end{array} \right\}
$$

Now let's demonstrate that Eq. (3.12) holds when $r = s + 1$.

In Eq. (3.13), s is substituted for 2 and also by Eqs. (3.3), (3.4), we write
$\lambda(\widehat{\wp}(t_1))^{\omega_1} \otimes \lambda(\widehat{\wp}(t_2))^{\omega_2} \otimes ... \otimes \lambda(\widehat{\wp}(t_s))^{\omega_s} \otimes \lambda(\widehat{\wp}(t_{s+1}))^{\omega_{s+1}}$

$$
= \bigcup_{\substack{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k) \ (k=1,2,...,s) \\ [b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k) \ (k=1,2,...,s) \\ [c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k) \ (k=1,2,...,s)}} \left\{ \begin{array}{l} \left\{ \left[\prod_{k=1}^{s}(1-(1-a_{(t_k)}^L)^\lambda)^{\omega(t_k)}, \prod_{k=1}^{s}(1-(1-a_{(t_k)}^U)^\lambda)^{\omega(t_k)} \right] \right\} \\ \left\{ \left[1-\prod_{k=1}^{s}(1-(b_{(t_k)}^L)^\lambda)^{\omega(t_k)}, 1-\prod_{k=1}^{s}(1-(b_{(t_k)}^U)^\lambda)^{\omega(t_k)} \right] \right\} \\ \left\{ \left[1-\prod_{k=1}^{s}(1-(c_{(t_k)}^L)^\lambda)^{\omega(t_k)}, 1-\prod_{k=1}^{s}(1-(c_{(t_k)}^U)^\lambda)^{\omega(t_k)} \right] \right\} \end{array} \right\}
$$

$$
\otimes \bigcup_{\substack{[a_{(t_{s+1})}^L, a_{(t_{s+1})}^U] \in \widehat{\alpha}(t_{s+1}) \\ [b_{(t_{s+1})}^L, b_{(t_{s+1})}^U] \in \widehat{\beta}(t_{s+1}) \\ [c_{(t_{s+1})}^L, c_{(t_{s+1})}^U] \in \widehat{\gamma}(t_{s+1})}} \left\{ \begin{array}{l} \left\{ \left[(1-(1-a_{(t_{s+1})}^L)^\lambda)^{\omega(t_{s+1})}, (1-(1-a_{(t_{s+1})}^U)^\lambda)^{\omega(t_{s+1})} \right] \right\} \\ \left[1-(1-(b_{(t_{s+1})}^L)^\lambda)^{\omega(t_{s+1})}, 1-(1-(b_{(t_{s+1})}^U)^\lambda)^{\omega(t_{s+1})} \right] \\ \left[1-(1-(c_{(t_{s+1})}^L)^\lambda)^{\omega(t_{s+1})}, 1-(1-(c_{(t_{s+1})}^U)^\lambda)^{\omega(t_{s+1})} \right] \end{array} \right\}
$$

$$= \bigcup_{\substack{[a^L_{(t_k)},a^U_{(t_k)}]\in\widehat{\alpha}(t_k)\ (k=1,2,...,s+1)\\ [b^L_{(t_k)},b^U_{(t_k)}]\in\widehat{\beta}(t_k)\ (k=1,2,...,s+1)\\ [c^L_{(t_k)},c^U_{(t_k)}]\in\widehat{\gamma}(t_k)\ (k=1,2,...,s+1)}} \left\{ \begin{cases} \left[\prod_{k=1}^{s+1}(1-(1-a^L_{(t_k)})^\lambda)^{\omega(t_k)}, \prod_{k=1}^{s+1}(1-(1-a^U_{(t_k)})^\lambda)^{\omega(t_k)}\right]\\ \left[1-\prod_{k=1}^{s+1}(1-(b^L_{(t_k)})^\lambda)^{\omega(t_k)}, 1-\prod_{k=1}^{s+1}(1-(b^U_{(t_k)})^\lambda)^{\omega(t_k)}\right]\\ \left[1-\prod_{k=1}^{s+1}(1-(c^L_{(t_k)})^\lambda)^{\omega(t_k)}, 1-\prod_{k=1}^{s+1}(1-(c^U_{(t_k)})^\lambda)^{\omega(t_k)}\right] \end{cases} \right\}$$

Then we obtain that (from Definition 3.2 (c))

$$\frac{1}{\lambda}(\lambda(\widehat{\wp}(t_1))^{\omega(t_1)} \otimes \lambda(\widehat{\wp}(t_2))^{\omega(t_2)} \otimes ... \otimes \lambda(\widehat{\wp})(t_s)^{\omega(t_s)} \otimes \lambda(\widehat{\wp}(t_{s+1}))^{\omega(t_{s+1})})$$

$$= \bigcup_{\substack{[a^L_{(t_k)},a^U_{(t_k)}]\in\widehat{\alpha}(t_k)\ (k=1,2,...,s+1)\\ [b^L_{(t_k)},b^U_{(t_k)}]\in\widehat{\beta}(t_k)\ (k=1,2,...,s+1)\\ [c^L_{(t_k)},c^U_{(t_k)}]\in\widehat{\gamma}(t_k)\ (k=1,2,...,s+1)}} \left\{ \begin{cases} \left[1-\left(1-\prod_{k=1}^{s+1}(1-(1-a^L_{(t_k)})^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, 1-\left(1-\prod_{k=1}^{s+1}(1-(1-a^U_{(t_k)})^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}\right]\\ \left[\left(1-\prod_{k=1}^{s+1}(1-(b^L_{(t_k)})^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, \left(1-\prod_{k=1}^{s+1}(1-(b^U_{(t_k)})^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}\right]\\ \left[\left(1-\prod_{k=1}^{s+1}(1-(c^L_{(t_k)})^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, \left(1-\prod_{k=1}^{s+1}(1-(c^U_{(t_k)})^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}\right] \end{cases} \right\}$$

$$= GDIVPHFWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_{s+1}))$$

Hence, Eq. (3.12) is valid for $r = s + 1$.

Thus, we conclude that Eq. (3.12) is true for all values of r. □

Note 2. If $\lambda = 1$ then GDIVPHFWA operator and GDIVPHFWG operator reduce to DIVPH-FWA operator and DIVPHFWG operator, respectively. Therefore, we will investigate the results of GDIPHFWA and GDIVPHFWG operators.

Theorem 3.7 *(Idempotency) Let $\widehat{\wp}(t_k)$ $(k = 1, 2, ..., r)$ be the IVPHFEs collected from k different periods t_k $(k = 1, 2, ..., r)$. If $\widehat{\wp}(t_k) = \widehat{\wp} = \{\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}\}$ for all $k = 1, 2, ..., r$ then*

(1) $GDIVPHFWA_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) = \widehat{\wp}$.
(2) $GDIVPHFWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_2)) = \widehat{\wp}$.

Proof Let us prove $GDIVPHFWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) = \widehat{\wp}$, the other can be shown similar to this.

Considering $\widehat{\wp}(t_k) = \widehat{\wp} = \{\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}\}$, by Theorem 3.6, we have

$$GDIVPHFWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r))$$

$$= \frac{1}{\lambda}(\lambda(\widehat{\wp}(t_1))^{\omega(t_1)} \otimes \lambda(\widehat{\wp}(t_2))^{\omega(t_2)} \otimes ... \otimes \lambda(\widehat{\wp}(t_r))^{\omega(t_r)})$$

$$= \bigcup_{[a^L,a^U]\in\widehat{\alpha},[b^L,b^U]\in\widehat{\beta},[c^L,c^U]\in\widehat{\gamma}} \left\{ \begin{cases} \left[1-\left(1-\prod_{k=1}^{r}(1-(1-a^L)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, 1-\left(1-\prod_{k=1}^{r}(1-(1-a^U)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}\right]\\ \left[\left(1-\prod_{k=1}^{r}(1-(b^L)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, \left(1-\prod_{k=1}^{r}(1-(b^U)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}\right]\\ \left[\left(1-\prod_{k=1}^{r}(1-(c^L)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, \left(1-\prod_{k=1}^{r}(1-(c^U)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}\right] \end{cases} \right\}$$

$$= \bigcup_{[a^L,a^U]\in\widehat{\alpha},[b^L,b^U]\in\widehat{\beta},[c^L,c^U]\in\widehat{\gamma}} \left\{ \begin{cases} \left[1-((1-a^L)^\lambda)^{\frac{1}{\lambda}}, 1-((1-a^U)^\lambda)^{\frac{1}{\lambda}}\right]\\ \left[((b^L)^\lambda)^{\frac{1}{\lambda}}, ((b^U)^\lambda)^{\frac{1}{\lambda}}\right]\\ \left[((c^L)^\lambda)^{\frac{1}{\lambda}}, ((c^U)^\lambda)^{\frac{1}{\lambda}}\right] \end{cases} \right\}$$

$$= \bigcup_{[a^L,a^U]\in\widehat{\alpha},[b^L,b^U]\in\widehat{\beta},[c^L,c^U]\in\widehat{\gamma}} \left\{ \begin{cases} \{[a^L, a^U]\}\\ \{[b^L, b^U]\}\\ \{[c^L, c^U]\} \end{cases} \right\}.$$

Thereby, the proof is completed. □

Theorem 3.8 *(Boundedness) Let $\widehat{\wp}(t_k)$ $(k = 1, 2, ..., r)$ be the IVPHFEs collected from k different periods t_k $(k = 1, 2, ..., r)$.*

(1) $\widehat{\wp}_{min} \leq GDIVPHFWA_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) \leq \widehat{\wp}_{max}$
(2) $\widehat{\wp}_{min} \leq GDIVPHFWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) \leq \widehat{\wp}_{max}$

where

$$\widehat{\wp}_{min} = \{\widehat{\alpha}_{min}, \widehat{\beta}_{max}, \widehat{\gamma}_{max}\} = \{\{[a^L_{min}, a^U_{min}]\}, \{[b^L_{max}, b^U_{max}]\}, \{[c^L_{max}, c^U_{max}]\}\},$$
$$\widehat{\wp}_{max} = \{\widehat{\alpha}_{max}, \widehat{\beta}_{min}, \widehat{\gamma}_{min}\} = \{\{[a^L_{max}, a^U_{max}]\}, \{[b^L_{min}, b^U_{min}]\}, \{[c^L_{min}, c^U_{min}]\}\}$$

for

$$a^L_{min} = \bigcup_{[a^L_{(t_k)}, a^U_{(t_k)}] \in \widehat{\alpha}(t_k)} min\{a^L_{(t_k)}\}, a^U_{min} = \bigcup_{[a^L_{(t_k)}, a^U_{(t_k)}] \in \widehat{\alpha}(t_k)} min\{a^U_{(t_k)}\},$$

$$a^L_{max} = \bigcup_{[a^L_{(t_k)}, a^U_{(t_k)}] \in \widehat{\alpha}(t_k)} max\{a^L_{(t_k)}\},$$

$$a^U_{max} = \bigcup_{[a^L_{(t_k)}, a^U_{(t_k)}] \in \widehat{\alpha}(t_k)} max\{a^U\}, b^L_{min} = \bigcup_{[b^L_{(t_k)}, b^U_{(t_k)}] \in \widehat{\beta}(t_k)} min\{b^L_{(t_k)}\},$$

$$b^U_{min} = \bigcup_{[b^L_{(t_k)}, b^U_{(t_k)}] \in \widehat{\beta}(t_k)} min\{b^U_{(t_k)}\},$$

$$b^L_{max} = \bigcup_{[b^L_{(t_k)}, b^U_{(t_k)}] \in \widehat{\beta}(t_k)} max\{b^L_{(t_k)}\}, b^U_{max} = \bigcup_{[b^L_{(t_k)}, b^U_{(t_k)}] \in \widehat{\beta}(t_k)} max\{b^U_{(t_k)}\},$$

$$c^L_{min} = \bigcup_{[c^L_{(t_k)}, c^U_{(t_k)}] \in \widehat{\gamma}(t_k)} min\{c^L_{(t_k)}\},$$

$$c^U_{min} = \bigcup_{[c^L_{(t_k)}, c^U_{(t_k)}] \in \widehat{\gamma}(t_k)} min\{c^U_{(t_k)}\}, c^L_{max} = \bigcup_{[c^L_{(t_k)}, c^U_{(t_k)}] \in \widehat{\gamma}(t_k)} max\{c^L_{(t_k)}\},$$

$$c^U_{max} = \bigcup_{[c^L_{(t_k)}, c^U_{(t_k)}] \in \widehat{\gamma}(t_k)} max\{c^U_{(t_k)}\}.$$

Proof Let us prove (2), the others can be demonstrate similar to this.
Since $a^L_{min} \leq a^L_{(t_k)} \leq a^L_{max}$, then

$$1 - a^L_{(t_k)} \leq 1 - a^L_{min} \Leftrightarrow 1 - (1 - a^L_{(t_k)})^\lambda \geq 1 - (1 - a^L_{min})^\lambda \Leftrightarrow \left(1 - \left(1 - a^L_{(t_k)}\right)^\lambda\right)^{\omega(t_k)}$$

$$\geq \left(1 - \left(1 - a^L_{min}\right)^\lambda\right)^{\omega(t_k)}$$

$$\Leftrightarrow \prod_{k=1}^{r} \left(1 - \left(1 - a^L_{(t_k)}\right)^\lambda\right)^{\omega(t_k)} \geq \prod_{k=1}^{r} \left(1 - \left(1 - a^L_{min}\right)^\lambda\right)^{\omega(t_k)}$$

$$\Leftrightarrow \left(1 - \prod_{k=1}^{r} \left(1 - \left(1 - a^L_{(t_k)}\right)^\lambda\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}$$

$$\leq \left(1 - \prod_{k=1}^{r}\left(1 - \left(1 - a_{\min}^{L}\right)^{\lambda}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}$$

$$\Leftrightarrow 1 - \left(1 - \prod_{k=1}^{r}\left(1 - \left(1 - a_{(t_k)}^{L}\right)^{\lambda}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}$$

$$\geq 1 - \left(1 - \prod_{k=1}^{r}\left(1 - \left(1 - a_{\min}^{L}\right)^{\lambda}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} = a_{\min}^{L}.$$

Similarly, we have

$$1 - \left(1 - \prod_{k=1}^{r}\left(1 - \left(1 - a_{(t_k)}^{L}\right)^{\lambda}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \leq 1 - \left(1 - \prod_{k=1}^{r}\left(1 - \left(1 - a_{\max}^{L}\right)^{\lambda}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} = a_{\max}^{L}.$$

Likewise, since $a_{\min}^{U} \leq a_{(t_k)}^{U} \leq a_{\max}^{U}$, we obtain that

$$1 - \left(1 - \prod_{k=1}^{r}\left(1 - \left(1 - a_{(t_k)}^{U}\right)^{\lambda}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \geq 1 - \left(1 - \prod_{k=1}^{r}\left(1 - \left(1 - a_{\min}^{U}\right)^{\lambda}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} = a_{\min}^{U}.$$

and

$$1 - \left(1 - \prod_{k=1}^{r}\left(1 - \left(1 - a_{(t_k)}^{U}\right)^{\lambda}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \leq 1 - \left(1 - \prod_{k=1}^{r}\left(1 - \left(1 - a_{\max}^{U}\right)^{\lambda}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} = a_{\max}^{U}.$$

Considering $b_{\min}^{U} \leq b_{(t_k)}^{U} \leq b_{\max}^{U}$, we found

$$b_{(t_k)}^{U} \leq b_{\max}^{U} \Leftrightarrow \left(b_{(t_k)}^{U}\right)^{\lambda} \leq \left(b_{\max}^{U}\right)^{\lambda} \Leftrightarrow 1 - \left(b_{(t_k)}^{U}\right)^{\lambda}$$

$$\geq 1 - \left(b_{\max}^{U}\right)^{\lambda} \Leftrightarrow \left(1 - \left(b_{(t_k)}^{U}\right)^{\lambda}\right)^{\omega(t_k)} \geq \left(1 - \left(b_{\max}^{U}\right)^{\lambda}\right)^{\omega(t_k)}$$

$$\Leftrightarrow \prod_{k=1}^{r}\left(1 - \left(b_{(t_k)}^{U}\right)^{\lambda}\right)^{\omega(t_k)} \geq \prod_{k=1}^{r}\left(1 - \left(b_{\max}^{U}\right)^{\lambda}\right)^{\omega(t_k)} \Leftrightarrow 1 - \prod_{k=1}^{r}\left(1 - \left(b_{(t_k)}^{U}\right)^{\lambda}\right)^{\omega(t_k)}$$

$$\leq 1 - \prod_{k=1}^{r}\left(1 - \left(b_{\max}^{U}\right)^{\lambda}\right)^{\omega(t_k)}$$

$$\times \left(1 - \prod_{k=1}^{r}\left(1 - \left(b_{(t_k)}^{U}\right)^{\lambda}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \leq \left(1 - \prod_{k=1}^{r}\left(1 - \left(b_{\max}^{U}\right)^{\lambda}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} = b_{\max}^{U}.$$

Similarly, we have

$$\left(1 - \prod_{k=1}^{r}\left(1 - \left(b_{(t_k)}^{U}\right)^{\lambda}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \geq \left(1 - \prod_{k=1}^{r}(1 - (b_{\min}^{U})^{\lambda})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} = b_{\min}^{U}.$$

Likewise, since $b_{\min}^{L} \leq b_{(t_k)}^{L} \leq b_{\max}^{L}$, we obtain that

$$b_{\min}^{L} = \left(1 - \prod_{k=1}^{r}\left(1 - \left(b_{\min}^{L}\right)^{\lambda}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \leq \left(1 - \prod_{k=1}^{r}\left(1 - \left(b_{(t_k)}^{L}\right)^{\lambda}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}$$

$$\leq \left(1 - \prod_{k=1}^{r}\left(1 - \left(b_{\max}^L\right)^\lambda\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} = b_{\max}^L.$$

For $c_{\min}^U \leq c_{(t_k)}^U \leq c_{\max}^U$ and $c_{\min}^L \leq c_{(t_k)}^L \leq c_{\max}^L$, we have

$$c_{\min}^L \leq \left(1 - \prod_{k=1}^{r}(1 - (c_{(t_k)}^L)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \leq c_{\max}^L, \text{ and } c_{\min}^U \leq \left(1 - \prod_{k=1}^{r}(1 - (c_{(t_k)}^U)^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \leq c_{\max}^U.$$

Let $GIVPHFWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) = \widehat{\wp} = \{\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}\}$ (from Theorem 3.6), then we say that

$$\mathcal{S}(\widehat{\wp}) = \frac{1 + \frac{1}{2}\left(\frac{1}{v(\widehat{\alpha})}\sum_{[a^L,a^U]\in\widehat{\alpha}}(a^L + a^U) - \frac{1}{v(\widehat{\beta})}\sum_{[b^L,b^U]\in\widehat{\beta}}(b^L + b^U) - \frac{1}{v(\widehat{\gamma})}\sum_{[c^L,c^U]\in\widehat{\gamma}}(c^L + c^U)\right)}{2}$$

$$\geq \frac{1 + \frac{1}{2}\left(\frac{1}{v(\widehat{\alpha}_{\min})}\sum_{[a_{\min}^L,a_{\min}^U]\in\widehat{\alpha}_{\min}}(a_{\min}^L + a_{\min}^U) - \frac{1}{v(\widehat{\beta}_{\min})}\sum_{[b_{\min}^L,b_{\min}^U]\in\widehat{\beta}_{\min}}(b_{\min}^L + b_{\min}^U) - \frac{1}{v(\widehat{\gamma}_{\min})}\sum_{[c_{\min}^L,c_{\min}^U]\in\widehat{\gamma}_{\min}}(c_{\min}^L + c_{\min}^U)\right)}{2} = \mathcal{S}(\widehat{\wp}_{\min})$$

$$\mathcal{S}(\widehat{\wp}) = \frac{1 + \frac{1}{2}\left(\frac{1}{v(\widehat{\alpha})}\sum_{[a^L,a^U]\in\widehat{\alpha}}(a^L + a^U) - \frac{1}{v(\widehat{\beta})}\sum_{[b^L,b^U]\in\widehat{\beta}}(b^L + b^U) - \frac{1}{v(\widehat{\gamma})}\sum_{[c^L,c^U]\in\widehat{\gamma}}(c^L + c^U)\right)}{2}$$

$$\leq \frac{1 + \frac{1}{2}\left(\frac{1}{v(\widehat{\alpha}_{\max})}\sum_{[a_{\max}^L,a_{\max}^U]\in\widehat{\alpha}_{\max}}(a_{\max}^L + a_{\max}^U) - \frac{1}{v(\widehat{\beta}_{\max})}\sum_{[b_{\max}^L,b_{\max}^U]\in\widehat{\beta}_{\max}}(b_{\max}^L + b_{\max}^U) - \frac{1}{v(\widehat{\gamma}_{\max})}\sum_{[c_{\max}^L,c_{\max}^U]\in\widehat{\gamma}_{\max}}(c_{\max}^L + c_{\max}^U)\right)}{2} = \mathcal{S}(\widehat{\wp}_{\max})$$

Then, we obtain $\widehat{\wp}_{\min} \leq GDIVPHFWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) \leq \widehat{\wp}_{\max}$. Thus, the boundary property is verified. □

Theorem 3.9 *(Monotonicity) Let* $(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r))$ *and* $(\widehat{\wp}^*(t_1), \widehat{\wp}^*(t_2), ..., \widehat{\wp}^*(t_r))$ *be two collections of the IVPHFEs collected from k different periods* t_k $(k = 1, 2, ..., r)$. *If* $\widehat{\wp}(t_k) \leq \widehat{\wp}^*(t_k)$ *for all* $k = 1, 2, ..., r$ *then*

1. $GDIVPHFWA_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) \leq GDIVPHFWA_{\omega(t)}(\widehat{\wp}^*(t_1), \widehat{\wp}^*(t_2), ..., \widehat{\wp}^*(t_r))$.
2. $GDIVPHFWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) \leq GDIVPHFWG_{\omega(t)}(\widehat{\wp}^*(t_1), \widehat{\wp}^*(t_2), ..., \widehat{\wp}^*(t_r))$.

Proof By considering Eqs. (3.11) and (3.12), they can be interpreted similar to the proof of Theorem 4.8. □

Theorem 3.10 *(Commutativity) Let* $(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r))$ *be a collections of IVPHFEs collected from k different periods* t_k $(k = 1, 2, ..., r)$. *In addition, let* $\widehat{\wp_n}(t_{1_n}), \widehat{\wp_n}(t_{2_n}), ..., \widehat{\wp_n}(t_{r_n})$ *be a new permutation of* $\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)$ *then*

1. $GDIVPHFWA_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) = GDIVPHFWA_{\omega(t)}(\widehat{\wp_n}(t_{1_n}), \widehat{\wp_n}(t_{2_n}), ..., \widehat{\wp_n}(t_{r_n}))$.
2. $GDIVPHFWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) = GDIVPHFWG_{\omega(t)}(\widehat{\wp_n}(t_{1_n}), \widehat{\wp_n}(t_{2_n}), ..., \widehat{\wp_n}(t_{r_n}))$.

Proof It can be easily seen by Theorem 3.6 (3) and (4). □

4 Dynamic interval-valued picture hesitant fuzzy weighted aggregation operators based on Einstein operations

In this part, we derive the dynamic interval-valued picture hesitant fuzzy weighted aggregation operators based on Einstein operations to fuse interval-valued picture hesitant fuzzy tools acquired at different time-moments.

Einstein operations include the Einstein sum and Einstein products (Klement et al. 2004). Einstein sum $\oplus_\epsilon$ is a t-conorm and Einstein product $\otimes_\epsilon$ is a t-norm, where for all $(x, y) \in [0, 1] \times [0, 1]$

$$x \oplus_\epsilon y = \frac{x + y}{1 + x + y} \quad \text{and} \quad x \otimes_\epsilon y = \frac{x + y}{1 + (1 - x)(1 - y)}.$$

Definition 4.1 Let $\widehat{\wp}(t_1) = \{\widehat{\alpha}(t_1), \widehat{\beta}(t_1), \widehat{\gamma}(t_1)\}$ and $\widehat{\wp}(t_2) = \{\widehat{\alpha}(t_2), \widehat{\beta}(t_2), \widehat{\gamma}(t_2)\}$ be two IVPHFEs and σ be any positive real number (i.e., $\sigma > 0$). Then, the Einstein operations of IVPHFEs are described as

(a)

$$\widehat{\wp}(t_1) \oplus_\epsilon \widehat{\wp}(t_2)$$

$$= \bigcup_{\substack{[a_{(t_1)}^L, a_{(t_1)}^U] \in \widehat{\alpha}(t_1),\ [a_{(t_2)}^L, a_{(t_2)}^U] \in \widehat{\alpha}(t_2) \\ [b^L(t_1), b^U(t_1)] \in \widehat{\beta}(t_1),\ [b_{(t_2)}^L, b_{(t_2)}^U] \in \widehat{\beta}(t_2) \\ [c_{(t_1)}^L, c_{(t_1)}^U] \in \widehat{\gamma}(t_1),\ [c_{(t_2)}^L, c_{(t_2)}^U] \in \widehat{\gamma}(t_2)}} \left\{ \left\{ \begin{bmatrix} \left[\frac{a_{(t_1)}^L + a_{(t_2)}^L}{1 + a_{(t_1)}^L a_{(t_2)}^L}, \frac{a_{(t_1)}^U + a_{(t_2)}^U}{1 + a_{(t_1)}^U a_{(t_2)}^U} \right] \\ \left[\frac{b_{(t_1)}^L b_{(t_2)}^L}{1 + (1 - b_{(t_1)}^L)(1 - b_{(t_2)}^L)}, \frac{b_{(t_1)}^U b_{(t_2)}^U}{1 + (1 - b_{(t_1)}^U)(1 - b_{(t_2)}^U)} \right] \\ \left[\frac{c_{(t_1)}^L c_{(t_2)}^L}{1 + (1 - c_{(t_1)}^L)(1 - c_{(t_2)}^L)}, \frac{c_{(t_1)}^U c_{(t_2)}^U}{1 + (1 - c_{(t_1)}^U)(1 - c_{(t_2)}^U)} \right] \end{bmatrix} \right\} \right\}.$$

(4.1)

(b)

$$\widehat{\wp}(t_1) \otimes_\epsilon \widehat{\wp}(t_2)$$

$$= \bigcup_{\substack{[a_{(t_1)}^L, a_{(t_1)}^U] \in \widehat{\alpha}(t_1),\ [a_{(t_2)}^L, a_{(t_2)}^U] \in \widehat{\alpha}(t_2) \\ [b_{(t_1)}^L, b_{(t_1)}^U] \in \widehat{\beta}(t_1),\ [b_{(t_2)}^L, b_{(t_2)}^U] \in \widehat{\beta}(t_2) \\ [c_{(t_1)}^L, c_{(t_1)}^U] \in \widehat{\gamma}(t_1),\ [c_{(t_2)}^L, c_{(t_2)}^U] \in \widehat{\gamma}(t_2)}} \left\{ \left\{ \begin{bmatrix} \left[\frac{a_{(t_1)}^L a_{(t_2)}^L}{1 + (1 - a_{(t_1)}^L)(1 - a_{(t_2)}^L)}, \frac{a_{(t_1)}^U a_{(t_2)}^U}{1 + (1 - a_{(t_1)}^U)(1 - a_{(t_2)}^U)} \right] \\ \left[\frac{b_{(t_1)}^L + b_{(t_2)}^L}{1 + b_{(t_1)}^L b_{(t_2)}^L}, \frac{b_{(t_1)}^U + b_{(t_2)}^U}{1 + b_{(t_1)}^U b_{(t_2)}^U} \right] \\ \left[\frac{c_{(t_1)}^L + c_{(t_2)}^L}{1 + c_{(t_1)}^L c_{(t_2)}^L}, \frac{c_{(t_1)}^U + c_{(t_2)}^U}{1 + c_{(t_1)}^U c_{(t_2)}^U} \right] \end{bmatrix} \right\} \right\}.$$

(4.2)

(c)

$$\sigma \widehat{\wp}(t_1) = \bigcup_{\substack{[a_{(t_1)}^L, a_{(t_1)}^U] \in \widehat{\alpha}(t_1) \\ [b_{(t_1)}^L, b_{(t_1)}^U] \in \widehat{\beta}(t_1) \\ [c_{(t_1)}^L, c_{(t_1)}^U] \in \widehat{\gamma}(t_1)}} \left\{ \left\{ \begin{bmatrix} \left[\frac{(1 + a_{(t_1)}^L)^m - (1 - a_{(t_1)}^L)^m}{(1 + a_{(t_1)}^L)^m + (1 - a_{(t_1)}^L)^m}, \frac{(1 + a_{(t_1)}^U)^m - (1 - a_{(t_1)}^U)^m}{(1 + a_{(t_1)}^U)^m + (1 - a_{(t_1)}^U)^m} \right] \\ \left[\frac{2(b_{(t_1)}^L)^m}{(2 - b_{(t_1)}^L)^m + (b_{(t_1)}^L)^m}, \frac{2(b_{(t_1)}^U)^m}{(2 - b_{(t_1)}^U)^m + (b_{(t_1)}^U)^m} \right] \\ \left[\frac{2(c_{(t_1)}^L)^m}{(2 - c_{(t_1)}^L)^m + (c_{(t_1)}^L)^m}, \frac{2(c_{(t_1)}^U)^m}{(2 - c_{(t_1)}^U)^m + (c_{(t_1)}^U)^m} \right] \end{bmatrix} \right\} \right\}. \quad (4.3)$$

(d)

$$(\widehat{\wp}(t_1))^\sigma = \bigcup_{\substack{[a_{(t_1)}^L, a_{(t_1)}^U] \in \widehat{\alpha}(t_1) \\ [b_{(t_1)}^L, b_{(t_1)}^U] \in \widehat{\beta}(t_1) \\ [c_{(t_1)}^L, c_{(t_1)}^U] \in \widehat{\gamma}(t_1)}} \left\{ \left\{ \begin{bmatrix} \left[\frac{2(a_{(t_1)}^L)^m}{(2 - a_{(t_1)}^L)^m + (a_{(t_1)}^L)^m}, \frac{2(a_{(t_1)}^U)^m}{(2 - a_{(t_1)}^U)^m + (a_{(t_1)}^U)^m} \right] \\ \left[\frac{(1 + b_{(t_1)}^L)^m - (1 - b_{(t_1)}^L)^m}{(1 + b_{(t_1)}^L)^m + (1 - b_{(t_1)}^L)^m}, \frac{(1 + b_{(t_1)}^U)^m - (1 - b_{(t_1)}^U)^m}{(1 + b_{(t_1)}^U)^m + (1 - b_{(t_1)}^U)^m} \right] \\ \left[\frac{(1 + c_{(t_1)}^L)^m - (1 - c_{(t_1)}^L)^m}{(1 + c_{(t_1)}^L)^m + (1 - c_{(t_1)}^L)^m}, \frac{(1 + c_{(t_1)}^U)^m - (1 - c_{(t_1)}^U)^m}{(1 + c_{(t_1)}^U)^m + (1 - c_{(t_1)}^U)^m} \right] \end{bmatrix} \right\} \right\}. \quad (4.4)$$

Example 4.2 We consider $\widehat{\wp}(t_1)$ and $\widehat{\wp}(t_2)$ given in Example 3.3. Also let $\sigma = 2$. Then, we obtain

$\widehat{\wp}(t_1) \oplus \widehat{\wp}(t_2)$

$= \begin{cases} \{[0.2941, 0.475], [0.3883, 0.5188]\} \\ \{[0.0055, 0.0312], [0.0084, 0.0491], [0.0116, 0.0689], [0.0116, 0.0384], [0.0178, 0.0604], [0.0243, 0.0845]\} \\ \{[0, 0.0116]\} \end{cases}$,

$\widehat{\wp}(t_1) \otimes \widehat{\wp}(t_2)$

$= \begin{cases} \{[0.0116, 0.0338], [0.0184, 0.0397]\} \\ \{[0.198, 0.4285], [0.2463, 0.5116], [0.2941, 0.5909], [0.2941, 0.4716], [0.3398, 0.5504], [0.3846, 0.625]\} \\ \{[0.1, 0.2941]\} \end{cases}$,

$2\widehat{\wp}(t_1) = \{\{[0.3846, 0.6236], [0.5505, 0.6896]\}, \{[0.0055, 0.04], [0.0243, 0.0604]\}, \{[0.0055, 0.0243]\}\}$,

$\widehat{\wp}(t_2)^2 = \{\{[0.0055, 0.013]\}, \{[0.198, 0.3846], [0.2934, 0.5504], [0.3846, 0.6896]\}, \{[0, 0.198]\}\}$.

Proposition 4.3 *Let* $\widehat{\wp}_1 = \{\widehat{\alpha}_1, \widehat{\beta}_1, \widehat{\gamma}_1\}$ *and* $\widehat{\wp}_2 = \{\widehat{\alpha}_2, \widehat{\beta}_2, \widehat{\gamma}_2\}$ *be two IVPHFEs and* $\sigma > 0$.

1. $\widehat{\wp}(t_1) \oplus_\epsilon \widehat{\wp}(t_2) = \widehat{\wp}(t_2) \oplus_\epsilon \widehat{\wp}(t_1)$.
2. $\widehat{\wp}(t_1) \otimes_\epsilon \widehat{\wp}(t_2) = \widehat{\wp}(t_2) \otimes_\epsilon \widehat{\wp}(t_1)$.
3. $\sigma(\widehat{\wp}(t_1) \oplus_\epsilon \widehat{\wp}(t_2)) = \sigma\widehat{\wp}(t_1) \oplus_\epsilon \sigma\widehat{\wp}(t_2)$.
4. $(\widehat{\wp}(t_1) \otimes_\epsilon \widehat{\wp}(t_2))^\sigma = (\widehat{\wp}(t_1))^\sigma \otimes_\epsilon (\widehat{\wp}(t_2))^\sigma$.

Proof It is trivial from Definition 4.1. □

Considering the above Einstein operation laws of IVPHFEs, we develop the following Einstein weighted aggregation operators.

Definition 4.4 Let $\widehat{\wp}(t_k)$ $(k = 1, 2, ..., r)$ be the IVPHFEs collected from k different periods t_k $(k = 1, 2, ..., r)$. In addition, let $\omega(t) = (\omega(t_1), \omega(t_2), ..., \omega(t_r))^T$ be the weighted vector of t_k $(k = 1, 2, ..., r)$, with $\omega(t_k) \in [0, 1]$ and $\sum_{k=1}^{r} \omega(t_k) = 1$, and $\lambda > 0$.

(a) A dynamic interval-valued picture hesitant fuzzy Einstein weighted average (DIVPH-FEWA) operator is a mapping $DIVPHFEWA_\epsilon^{\omega(t)} : \mathcal{P}^r \to \mathcal{P}$, such that

$$DIVPHFEWA_\epsilon^{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r))$$

$$= \bigoplus_{k=1}^{r}{}_\epsilon \omega(t_k)\widehat{\wp}(t_k) = \omega(t_1)\widehat{\wp}(t_1) \oplus_\epsilon \omega(t_2)\widehat{\wp}(t_2) \oplus_\epsilon ... \oplus_\epsilon \omega(t_r)\widehat{\wp}(t_r) \quad (4.5)$$

(b) A dynamic interval-valued picture hesitant fuzzy Einstein weighted geometric (DIVPH-FEWG) operator is a mapping $DIVPHFEWG_\epsilon^{\omega(t)} : \mathcal{P}^r \to \mathcal{P}$, such that

$$DIVPHFEWG_\epsilon^{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r))$$

$$= \bigotimes_{k=1}^{r}{}_\epsilon (\widehat{\wp}(t_k))^{\omega(t_k)} = (\widehat{\wp}(t_1))^{\omega(t_1)} \otimes_\epsilon (\widehat{\wp}(t_2))^{\omega(t_2)} \otimes_\epsilon ... \otimes_\epsilon (\widehat{\wp}(t_r))^{\omega(t_r)} \quad (4.6)$$

(c) A generalized dynamic interval-valued picture hesitant fuzzy Einstein weighted average (GDIVPHFEWA) operator is a mapping $GDIVPHFEWA_\epsilon^{\omega(t)} : \mathcal{P}^r \to \mathcal{P}$, such that

$$GDIVPHFEWA_\epsilon^{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r))$$

$$= \left(\bigoplus_{k=1}^{r}{}_\epsilon \omega(t_k)(\widehat{\wp}(t_k))^\lambda \right)^{\frac{1}{\lambda}} = (\omega(t_1)(\widehat{\wp}(t_1))^\lambda \oplus_\epsilon \omega(t_2)(\widehat{\wp}(t_2))^\lambda \oplus_\epsilon ... \oplus_\epsilon \omega(t_k)(\widehat{\wp}(t_k))^\lambda)^{\frac{1}{\lambda}}$$

$$(4.7)$$

(d) A generalized dynamic interval-valued picture hesitant fuzzy Einstein weighted geometric (GDIVPHFEWG) operator is a mapping $GDIVPHFEWG_\epsilon^{\omega(t)} : \mathcal{P}^r \to \mathcal{P}$, such that

$$GDIVPHFEWG_\epsilon^{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r))$$

$$= \frac{1}{\lambda}\left(\bigotimes_{k=1}^{r}{}_\epsilon \ \lambda(\widehat{\wp}(t_k))^{\omega(t_k)} \right) = \frac{1}{\lambda}(\lambda(\widehat{\wp}(t_1))^{\omega(t_1)} \otimes_\epsilon \lambda(\widehat{\wp}(t_2))^{\omega(t_2)} \otimes_\epsilon ... \otimes_\epsilon \lambda(\widehat{\wp}(t_r))^{\omega(t_r)})$$

$$(4.8)$$

Theorem 4.5 *Let $\widehat{\wp}(t_k)$ ($k = 1, 2, ..., r$) be the IVPHFEs collected from k different periods t_k ($k = 1, 2, ..., r$). Also, let $\omega(t) = (\omega(t_1), \omega(t_2), ..., \omega(t_r))^T$ be the weighted vector of t_k ($k = 1, 2, ..., r$), with $\omega(t_k) \in [0, 1]$ and $\sum\limits_{k=1}^{r} \omega(t_k) = 1$, and $\lambda > 0$.*

(1) The aggregation value of IVPHFEs by using the DIVPHFEWA operator is also an IVPHFE, and

$$DIVPHFEWA_\epsilon^{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r))$$

$$= \bigoplus_{k=1}^{r}{}_\epsilon \ \omega(t_k)\widehat{\wp}(t_k)$$

$$= \bigcup_{\substack{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k)\ (k=1,2,...,r) \\ [b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k)\ (k=1,2,...,r) \\ [c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k)\ (k=1,2,...,r)}} \left\{ \left\{ \begin{matrix} \left[\frac{\prod_{k=1}^r (1+a_{(t_k)}^L)^{\omega(t_k)} - \prod_{k=1}^r (1-a_{(t_k)}^L)^{\omega(t_k)}}{\prod_{k=1}^r (1+a_{(t_k)}^L)^{\omega(t_k)} + \prod_{k=1}^r (1-a_{(t_k)}^L)^{\omega(t_k)}}, \ \frac{\prod_{k=1}^r (1+a_{(t_k)}^U)^{\omega(t_k)} - \prod_{k=1}^r (1-a_{(t_k)}^U)^{\omega(t_k)}}{\prod_{k=1}^r (1+a_{(t_k)}^U)^{\omega(t_k)} + \prod_{k=1}^r (1-a_{(t_k)}^U)^{\omega(t_k)}} \right] \\ \left[\frac{2\prod_{k=1}^r (b_{(t_k)}^L)^{\omega(t_k)}}{\prod_{k=1}^r (2-b_{(t_k)}^L)^{\omega(t_k)} + \prod_{k=1}^r (b_{(t_k)}^L)^{\omega(t_k)}}, \ \frac{2\prod_{k=1}^r (b_{(t_k)}^U)^{\omega(t_k)}}{\prod_{k=1}^r (2-b_{(t_k)}^U)^{\omega(t_k)} + \prod_{k=1}^r (b_{(t_k)}^U)^{\omega(t_k)}} \right] \\ \left[\frac{2\prod_{k=1}^r (c_{(t_k)}^L)^{\omega(t_k)}}{\prod_{k=1}^r (2-c_{(t_k)}^L)^{\omega(t_k)} + \prod_{k=1}^r (c_{(t_k)}^L)^{\omega(t_k)}}, \ \frac{2\prod_{k=1}^r (c_{(t_k)}^U)^{\omega(t_k)}}{\prod_{k=1}^r (2-c_{(t_k)}^U)^{\omega(t_k)} + \prod_{k=1}^r (c_{(t_k)}^U)^{\omega(t_k)}} \right] \end{matrix} \right\} \right\}$$

$$(4.9)$$

(2) The aggregation value of IVPHFEs by using the DIVPHFEWG operator is also an IVPHFE, and

$$DIVPHFEWG_\epsilon^{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r))$$

$$= \bigotimes_{k=1}^{r}{}_\epsilon \ (\widehat{\wp}(t_k))^{\omega(t_k)}$$

$$= \bigcup_{\substack{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k)(k=1,2,...,r) \\ [b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k)(k=1,2,...,r) \\ [c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k)(k=1,2,...,r)}} \left\{ \left\{ \begin{matrix} \left[\frac{2\prod_{k=1}^r (a_{(t_k)}^L)^{\omega(t_k)}}{\prod_{k=1}^r (2-a_{(t_k)}^L)^{\omega(t_k)} + \prod_{k=1}^r (a_{(t_k)}^L)^{\omega(t_k)}}, \ \frac{2\prod_{k=1}^r (a_{(t_k)}^U)^{\omega(t_k)}}{\prod_{k=1}^r (2-a_{(t_k)}^U)^{\omega(t_k)} + \prod_{k=1}^r (a_{(t_k)}^U)^{\omega(t_k)}} \right] \\ \left[\frac{\prod_{k=1}^r (1+b_{(t_k)}^L)^{\omega(t_k)} - \prod_{k=1}^r (1-b_{(t_k)}^L)^{\omega(t_k)}}{\prod_{k=1}^r (1+b_{(t_k)}^L)^{\omega(t_k)} + \prod_{k=1}^r (1-b_{(t_k)}^L)^{\omega(t_k)}}, \ \frac{\prod_{k=1}^r (1+b_{(t_k)}^U)^{\omega(t_k)} - \prod_{k=1}^r (1-b_{(t_k)}^U)^{\omega(t_k)}}{\prod_{k=1}^r (1+b_{(t_k)}^U)^{\omega(t_k)} + \prod_{k=1}^r (1-b_{(t_k)}^U)^{\omega(t_k)}} \right] \\ \left[\frac{\prod_{k=1}^r (1+c_{(t_k)}^L)^{\omega(t_k)} - \prod_{k=1}^r (1-c_{(t_k)}^L)^{\omega(t_k)}}{\prod_{k=1}^r (1+c_{(t_k)}^L)^{\omega(t_k)} + \prod_{k=1}^r (1-c_{(t_k)}^L)^{\omega(t_k)}}, \ \frac{\prod_{k=1}^r (1+c_{(t_k)}^U)^{\omega(t_k)} - \prod_{k=1}^r (1-c_{(t_k)}^U)^{\omega(t_k)}}{\prod_{k=1}^r (1+c_{(t_k)}^U)^{\omega(t_k)} + \prod_{k=1}^r (1-c_{(t_k)}^U)^{\omega(t_k)}} \right] \end{matrix} \right\} \right\}$$

$$(4.10)$$

(3) The aggregation value of IVPHFEs by using the GDIVPHFEWA operator is also an IVPHFE, and

$$GDIVPHFEWA_\epsilon^{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r))$$

$$= (\bigoplus_{k=1}^{r}{}_\epsilon \ \omega(t_k)(\widehat{\wp}(t_k))^\lambda)^{\frac{1}{\lambda}}$$

$$
= \bigcup_{\substack{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k)(k=1,2,\dots,r) \\ [b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k)(k=1,2,\dots,r) \\ [c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k)(k=1,2,\dots,r)}}
\left\{ \left\{
\left[
\frac{2\left(\prod_{k=1}^{r}(\eta_{a_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^{r}(\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}(\eta_{a_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}(\eta_{a_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^{r}(\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}},
\right.\right.\right.
$$

$$
\left.
\frac{2\left(\prod_{k=1}^{r}(\eta_{a_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^{r}(\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}(\eta_{a_{(t_k)}^U})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}(\eta_{a_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^{r}(\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}
\right],
$$

$$
\left[
\frac{\left(\prod_{k=1}^{r}(\zeta_{b_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\theta_{b_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{r}(\zeta_{b_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^{r}(\theta_{b_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}(\zeta_{b_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\theta_{b_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}(\zeta_{b_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^{r}(\theta_{b_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}},
\right.
$$

$$
\left.
\frac{\left(\prod_{k=1}^{r}(\zeta_{b_{(t_k)}^U})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\theta_{b_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{r}(\zeta_{b_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^{r}(\theta_{b_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}(\zeta_{b_{(t_k)}^U})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\theta_{b_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}(\zeta_{b_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^{r}(\theta_{b_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}
\right],
$$

$$
\left[
\frac{\left(\prod_{k=1}^{r}(\xi_{c_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\psi_{c_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{r}(\xi_{c_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^{r}(\psi_{c_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}(\xi_{c_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\psi_{c_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}(\xi_{c_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^{r}(\psi_{c_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}},
\right.
$$

$$
\left.\left.\left.
\frac{\left(\prod_{k=1}^{r}(\xi_{c_{(t_k)}^U})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\psi_{c_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{r}(\xi_{c_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^{r}(\psi_{c_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}(\xi_{c_{(t_k)}^U})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\psi_{c_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}(\xi_{c_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^{r}(\psi_{c_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}
\right]\right]\right\}\right\}
$$

$$(4.11)$$

where

$$
\begin{cases}
\eta_{a_{(t_k)}^L} = \left(2 - a_{(t_k)}^L\right)^{\lambda} + 3\left(a_{(t_k)}^L\right)^{\lambda}, \quad \vartheta_{a_{(t_k)}^L} = \left(2 - a_{(t_k)}^L\right)^{\lambda} - \left(a_{(t_k)}^L\right)^{\lambda}, \\
\eta_{a_{(t_k)}^U} = \left(2 - a_{(t_k)}^U\right)^{\lambda} + 3\left(a_{(t_k)}^U\right)^{\lambda}, \quad \vartheta_{a_{(t_k)}^U} = \left(2 - a_{(t_k)}^U\right)^{\lambda} - \left(a_{(t_k)}^U\right)^{\lambda}, \\
\zeta_{b_{(t_k)}^L} = \left(1 + b_{(t_k)}^L\right)^{\lambda} + 3\left(1 - b_{(t_k)}^L\right)^{\lambda}, \quad \theta_{b_{(t_k)}^L} = \left(1 + b_{(t_k)}^L\right)^{\lambda} - \left(1 - b_{(t_k)}^L\right)^{\lambda}, \\
\zeta_{b_{(t_k)}^U} = \left(1 + b_{(t_k)}^U\right)^{\lambda} + 3\left(1 - b_{(t_k)}^U\right)^{\lambda}, \quad \theta_{b_{(t_k)}^U} = \left(1 + b_{(t_k)}^U\right)^{\lambda} - \left(1 - b_{(t_k)}^U\right)^{\lambda}, \\
\xi_{c_{(t_k)}^L} = \left(1 + c_{(t_k)}^L\right)^{\lambda} + 3\left(1 - c_{(t_k)}^L\right)^{\lambda}, \quad \psi_{c_{(t_k)}^L} = \left(1 + c_{(t_k)}^L\right)^{\lambda} - \left(1 - c_{(t_k)}^L\right)^{\lambda}, \\
\xi_{c_{(t_k)}^U} = \left(1 + c_{(t_k)}^U\right)^{\lambda} + 3\left(1 - c_{(t_k)}^U\right)^{\lambda}, \quad \psi_{c_{(t_k)}^U} = \left(1 + c_{(t_k)}^U\right)^{\lambda} - \left(1 - c_{(t_k)}^U\right)^{\lambda}.
\end{cases}
$$

$$(4.12)$$

4. *The aggregation value of IVPHFEs by using the GDIVPHFEWG operator is also an IVPHFE, and*

$$
GDIVPHFEWG_\epsilon^{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), \dots, \widehat{\wp}(t_r))
$$

$$
= \frac{1}{\lambda}\left(\bigotimes_{k=1}^{r}{}_\epsilon \lambda(\widehat{\wp}(t_k))^{\omega(t_k)}\right)
$$

$$
= \bigcup_{\substack{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k)(k=1,2,\dots,r) \\ [b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k)(k=1,2,\dots,r) \\ [c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k)(k=1,2,\dots,r)}}
\left\{
\begin{array}{l}
\left\{
\left[
\frac{\left(\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^r (\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^r (\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^r (\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^r (\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}},
\right.\right. \\
\left.\left.
\frac{\left(\prod_{k=1}^r (\eta_{a_{(t_k)}^U})^{\omega(t_k)} + 3\prod_{k=1}^r (\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^r (\eta_{a_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^r (\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r (\eta_{a_{(t_k)}^U})^{\omega(t_k)} + 3\prod_{k=1}^r (\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^r (\eta_{a_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^r (\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}
\right]\right\} \\[4ex]
\left\{
\left[
\frac{2\left(\prod_{k=1}^r (\zeta_{b_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^r (\theta_{b_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r (\zeta_{b_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^r (\theta_{b_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^r (\zeta_{b_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^r (\theta_{b_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}},
\right.\right. \\
\left.\left.
\frac{2\left(\prod_{k=1}^r (\zeta_{b_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^r (\theta_{b_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r (\zeta_{b_{(t_k)}^U})^{\omega(t_k)} + 3\prod_{k=1}^r (\theta_{b_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^r (\zeta_{b_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^r (\theta_{b_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}
\right]\right\} \\[4ex]
\left\{
\left[
\frac{2\left(\prod_{k=1}^r (\xi_{c_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^r (\psi_{c_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r (\xi_{c_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^r (\psi_{c_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^r (\xi_{c_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^r (\psi_{c_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}},
\right.\right. \\
\left.\left.
\frac{2\left(\prod_{k=1}^r (\xi_{c_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^r (\psi_{c_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r (\xi_{c_{(t_k)}^U})^{\omega(t_k)} + 3\prod_{k=1}^r (\psi_{c_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^r (\xi_{c_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^r (\psi_{c_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}
\right]\right\}
\end{array}
\right\}
\tag{4.13}
$$

where

$$
\begin{cases}
\eta_{a_{(t_k)}^L} = \left(1 + a_{(t_k)}^L\right)^{\lambda} + 3\left(1 - a_{(t_k)}^L\right)^{\lambda}, & \vartheta_{a_{(t_k)}^L} = \left(1 + a_{(t_k)}^L\right)^{\lambda} - \left(1 - a_{(t_k)}^L\right)^{\lambda}, \\
\eta_{a_{(t_k)}^U} = \left(1 + a_{(t_k)}^U\right)^{\lambda} + 3\left(1 - a_{(t_k)}^U\right)^{\lambda}, & \vartheta_{a_{(t_k)}^U} = \left(1 + a_{(t_k)}^U\right)^{\lambda} - \left(1 - a_{(t_k)}^U\right)^{\lambda}, \\
\zeta_{b_{(t_k)}^L} = \left(2 - b_{(t_k)}^L\right)^{\lambda} + 3\left(b_{(t_k)}^L\right)^{\lambda}, & \theta_{b_{(t_k)}^L} = \left(2 - b_{(t_k)}^L\right)^{\lambda} - \left(b_{(t_k)}^L\right)^{\lambda}, \\
\zeta_{b_{(t_k)}^U} = \left(2 - b_{(t_k)}^U\right)^{\lambda} + 3\left(b_{(t_k)}^U\right)^{\lambda}, & \theta_{b_{(t_k)}^U} = \left(2 - b_{(t_k)}^U\right)^{\lambda} - \left(b_{(t_k)}^U\right)^{\lambda}, \\
\xi_{c_{(t_k)}^L} = \left(2 - c_{(t_k)}^L\right)^{\lambda} + 3\left(c_{(t_k)}^L\right)^{\lambda}, & \psi_{c_{(t_k)}^L} = \left(2 - c_{(t_k)}^L\right)^{\lambda} - \left(c_{(t_k)}^L\right)^{\lambda}, \\
\xi_{c_{(t_k)}^U} = \left(2 - c_{(t_k)}^U\right)^{\lambda} + 3\left(c_{(t_k)}^U\right)^{\lambda}, & \psi_{c_{(t_k)}^U} = \left(2 - c_{(t_k)}^U\right)^{\lambda} - \left(c_{(t_k)}^U\right)^{\lambda}.
\end{cases}
\tag{4.14}
$$

Proof Let us prove (3), the others can be demonstrate similar to this.

We use the induction technique for proof.

- For $r = 1$:

  Then we obtain the following

$$
GDIVPHFEWA_\epsilon^{\omega(t)}(\widehat{\wp}(t_1)) = (\omega(t_1)(\widehat{\wp}(t_1))^{\lambda})^{\frac{1}{\lambda}} = \widehat{\wp}(t_1).
$$

Because from Definition 4.1 (d) we calculate $(\widehat{\wp}(t_1))^{\lambda}$ as

$$
(\widehat{\wp}(t_1))^{\lambda} = \bigcup_{\substack{[a_{(t_1)}^L, a_{(t_1)}^U] \in \widehat{\alpha}(t_1) \\ [b_{(t_1)}^L, b_{(t_1)}^U] \in \widehat{\beta}(t_1) \\ [c_{(t_1)}^L, c_{(t_1)}^U] \in \widehat{\gamma}(t_1)}}
\left\{
\begin{array}{l}
\left\{\left[\frac{2(a_{(t_1)}^L)^{\lambda}}{(2-a_{(t_1)}^L)^{\lambda} + (a_{(t_1)}^L)^{\lambda}}, \frac{2(a_{(t_1)}^U)^{\lambda}}{(2-a_{(t_1)}^U)^{\lambda} + (a_{(t_1)}^U)^{\lambda}}\right]\right\} \\
\left\{\left[\frac{(1+b_{(t_1)}^L)^{\lambda} - (1-(b_{(t_1)}^L)^{\lambda}}{(1+b_{(t_1)}^L)^{\lambda} + (1-(b_{(t_1)}^L)^{\lambda}}, \frac{(1+b_{(t_1)}^U)^{\lambda} - (1-(b_{(t_1)}^U)^{\lambda}}{(1+b_{(t_1)}^U)^{\lambda} + (1-(b_{(t_1)}^U)^{\lambda}}\right]\right\} \\
\left\{\left[\frac{(1+c_{(t_1)}^L)^{\lambda} - (1-(c_{(t_1)}^L)^{\lambda}}{(1+c_{(t_1)}^L)^{\lambda} + (1-(c_{(t_1)}^L)^{\lambda}}, \frac{(1+c_{(t_1)}^U)^{\lambda} - (1-(c_{(t_1)}^U)^{\lambda}}{(1+c_{(t_1)}^U)^{\lambda} + (1-(c_{(t_1)}^U)^{\lambda}}\right]\right\}
\end{array}
\right\}.
$$

Since $\omega(t_1) = 1$, we have $\omega(t_1)(\widehat{\wp}(t_1))^\lambda = (\widehat{\wp}(t_1))^\lambda$ by Definition 4.1 (c). Then, we obtain that (from Definition 4.1 (d))

$$
(\omega(t_1)(\widehat{\wp}(t_1))^\lambda)^{\frac{1}{\lambda}}
$$

$$
= \bigcup_{\substack{[a^L_{(t_1)},a^U_{(t_1)}]\in\widehat{\alpha}(t_1) \\ [b^L_{(t_1)},b^U_{(t_1)}]\in\widehat{\beta}(t_1) \\ [c^L_{(t_1)},c^U_{(t_1)}]\in\widehat{\gamma}(t_1)}} \left\{ \left\{ \begin{array}{l} \left[\dfrac{2\left(\frac{2(a^L_{(t_1)})^\lambda}{(2-a^L_{(t_1)})^\lambda+(a^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}}{\left(2-\frac{2(a^L_{(t_1)})^\lambda}{(2-a^L_{(t_1)})^\lambda+(a^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}+\left(\frac{2(a^L_{(t_1)})^\lambda}{(2-a^L_{(t_1)})^\lambda+(a^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}}, \dfrac{2\left(\frac{2(a^U_{(t_1)})^\lambda}{(2-a^U_{(t_1)})^\lambda+(a^U_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}}{\left(2-\frac{2(a^U_{(t_1)})^\lambda}{(2-a^U_{(t_1)})^\lambda+(a^U_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}+\left(\frac{2(a^U_{(t_1)})^\lambda}{(2-a^U_{(t_1)})^\lambda+(a^U_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}} \right] \\[2em] \left[\dfrac{\left(1+\frac{(1+b^L_{(t_1)})^\lambda-(1-b^L_{(t_1)})^\lambda}{(1+b^L_{(t_1)})^\lambda+(1-b^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}-\left(1-\frac{(1+b^L_{(t_1)})^\lambda-(1-b^L_{(t_1)})^\lambda}{(1+b^L_{(t_1)})^\lambda+(1-b^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}}{\left(1+\frac{(1+b^L_{(t_1)})^\lambda-(1-b^L_{(t_1)})^\lambda}{(1+b^L_{(t_1)})^\lambda+(1-b^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}+\left(1-\frac{(1+b^L_{(t_1)})^\lambda-(1-b^L_{(t_1)})^\lambda}{(1+b^L_{(t_1)})^\lambda+(1-b^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}}, \dfrac{\left(1+\frac{(1+b^U_{(t_1)})^\lambda-(1-b^U_{(t_1)})^\lambda}{(1+b^U_{(t_1)})^\lambda+(1-b^U_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}-\left(1-\frac{(1+b^U_{(t_1)})^\lambda-(1-b^U_{(t_1)})^\lambda}{(1+b^U_{(t_1)})^\lambda+(1-b^U_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}}{\left(1+\frac{(1+b^U_{(t_1)})^\lambda-(1-b^U_{(t_1)})^\lambda}{(1+b^U_{(t_1)})^\lambda+(1-b^U_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}+\left(1-\frac{(1+b^U_{(t_1)})^\lambda-(1-b^U_{(t_1)})^\lambda}{(1+b^U_{(t_1)})^\lambda+(1-b^U_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}} \right] \\[2em] \left[\dfrac{\left(1+\frac{(1+c^L_{(t_1)})^\lambda-(1-c^L_{(t_1)})^\lambda}{(1+c^L_{(t_1)})^\lambda+(1-c^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}-\left(1-\frac{(1+c^L_{(t_1)})^\lambda-(1-c^L_{(t_1)})^\lambda}{(1+c^L_{(t_1)})^\lambda+(1-c^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}}{\left(1+\frac{(1+c^L_{(t_1)})^\lambda-(1-c^L_{(t_1)})^\lambda}{(1+c^L_{(t_1)})^\lambda+(1-c^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}+\left(1-\frac{(1+c^L_{(t_1)})^\lambda-(1-c^L_{(t_1)})^\lambda}{(1+c^L_{(t_1)})^\lambda+(1-c^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}}, \dfrac{\left(1+\frac{(1+c^U_{(t_1)})^\lambda-(1-c^U_{(t_1)})^\lambda}{(1+c^U_{(t_1)})^\lambda+(1-c^U_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}-\left(1-\frac{(1+c^U_{(t_1)})^\lambda-(1-c^U_{(t_1)})^\lambda}{(1+c^U_{(t_1)})^\lambda+(1-c^U_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}}{\left(1+\frac{(1+c^U_{(t_1)})^\lambda-(1-c^U_{(t_1)})^\lambda}{(1+c^U_{(t_1)})^\lambda+(1-c^U_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}+\left(1-\frac{(1+c^U_{(t_1)})^\lambda-(1-c^U_{(t_1)})^\lambda}{(1+c^U_{(t_1)})^\lambda+(1-c^U_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}} \right] \end{array} \right\} \right\}.
$$

We demonstrate that

$$
\frac{2\left(\frac{2(a^L_{(t_1)})^\lambda}{(2-a^L_{(t_1)})^\lambda+(a^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}}{\left(2-\frac{2(a^L_{(t_1)})^\lambda}{(2-a^L_{(t_1)})^\lambda+(a^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}+\left(\frac{2(a^L_{(t_1)})^\lambda}{(2-a^L_{(t_1)})^\lambda+(a^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}} = \frac{\frac{2^{\frac{\lambda+1}{\lambda}}((a^L_{(t_1)})^\lambda)^{\frac{1}{\lambda}}}{((2-a^L_{(t_1)})^\lambda+(a^L_{(t_1)})^\lambda)^{\frac{1}{\lambda}}}}{\frac{2^{\frac{1}{\lambda}}((2-a^L_{(t_1)})^\lambda)^{\frac{1}{\lambda}}+((a^L_{(t_1)})^\lambda)^{\frac{1}{\lambda}}}{((2-a^L_{(t_1)})^\lambda+(a^L_{(t_1)})^\lambda)^{\frac{1}{\lambda}}}}
$$

$$
= \frac{2a^L_{(t_1)}}{2} = a^L_{(t_1)}
$$

and

$$
\frac{\left(1+\frac{(1+b^L_{(t_1)})^\lambda-(1-b^L_{(t_1)})^\lambda}{(1+b^L_{(t_1)})^\lambda+(1-b^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}-\left(1-\frac{(1+b^L_{(t_1)})^\lambda-(1-b^L_{(t_1)})^\lambda}{(1+b^L_{(t_1)})^\lambda+(1-b^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}}{\left(1+\frac{(1+b^L_{(t_1)})^\lambda-(1-b^L_{(t_1)})^\lambda}{(1+b^L_{(t_1)})^\lambda+(1-b^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}+\left(1-\frac{(1+b^L_{(t_1)})^\lambda-(1-b^L_{(t_1)})^\lambda}{(1+b^L_{(t_1)})^\lambda+(1-b^L_{(t_1)})^\lambda}\right)^{\frac{1}{\lambda}}}
$$

$$
= \frac{\frac{\left(2(1+b^L_{(t_1)})^\lambda)^{\frac{1}{\lambda}}-(2(1-b^L_{(t_1)})^\lambda\right)^{\frac{1}{\lambda}}}{\left((1+b^L_{(t_1)})^\lambda-(1-b^L_{(t_1)})^\lambda\right)^{\frac{1}{\lambda}}}}{\frac{\left(2(1+b^L_{(t_1)})^\lambda)^{\frac{1}{\lambda}}-(2(1-b^L_{(t_1)})^\lambda\right)^{\frac{1}{\lambda}}}{\left((1+b^L_{(t_1)})^\lambda-(1-b^L_{(t_1)})^\lambda\right)^{\frac{1}{\lambda}}}} = \frac{2b^L_{(t_1)}}{2} = b^L_{(t_1)}
$$

and also others are demonstrated similarly.

Therefore, we have $GDIVPHFEWA_\epsilon^{\omega(t)}(\widehat{\wp}(t_1)) = (\omega(t_1)(\widehat{\wp}(t_1))^\lambda)^{\frac{1}{\lambda}} = \widehat{\wp}(t_1)$ and so Eq. (4.11) holds for $r = 1$.

- For $r = 2$:

  By considering Eq. (4.12) for $k = 1, 2$, we have (from Definition 4.1 (c) and (d))

$$
\omega(t_1)(\widehat{\wp}(t_1))^\lambda = \bigcup_{\substack{[a^L_{(t_1)}, a^U_{(t_1)}] \in \widehat{\alpha}(t_1) \\ [b^L_{(t_1)}, b^U_{(t_1)}] \in \widehat{\beta}(t_1) \\ [c^L_{(t_1)}, c^U_{(t_1)}] \in \widehat{\gamma}(t_1)}} \left\{ \left\{ \left[\frac{(\eta_{a^L_{(t_1)}})^{\omega(t_1)} - (\vartheta_{a^L_{(t_1)}})^{\omega(t_1)}}{(\eta_{a^L_{(t_1)}})^{\omega(t_1)} + (\vartheta_{a^L_{(t_1)}})^{\omega(t_1)}}, \frac{(\eta_{a^U_{(t_1)}})^{\omega(t_1)} - (\vartheta_{a^U_{(t_1)}})^{\omega(t_1)}}{(\eta_{a^U_{(t_1)}})^{\omega(t_1)} + (\vartheta_{a^U_{(t_1)}})^{\omega(t_1)}} \right] \right\}, \left[\frac{2(\theta_{b^L_{(t_1)}})^{\omega(t_1)}}{(\zeta_{b^L_{(t_1)}})^{\omega(t_1)} + (\theta_{b^L_{(t_1)}})^{\omega(t_1)}}, \frac{2(\theta_{b^U_{(t_1)}})^{\omega(t_1)}}{(\zeta_{b^U_{(t_1)}})^{\omega(t_1)} + (\theta_{b^U_{(t_1)}})^{\omega(t_1)}} \right], \left[\frac{2(\psi_{c^L_{(t_1)}})^{\omega(t_1)}}{(\xi_{c^L_{(t_1)}})^{\omega(t_1)} + (\psi_{c^L_{(t_1)}})^{\omega(t_1)}}, \frac{2(\psi_{c^U_{(t_1)}})^{\omega(t_1)}}{(\xi_{c^U_{(t_1)}})^{\omega(t_1)} + (\psi_{c^U_{(t_1)}})^{\omega(t_1)}} \right] \right\}
$$

and

$$
\omega(t_2)(\widehat{\wp}(t_2))^\lambda = \bigcup_{\substack{[a^L_{(t_2)}, a^U_{(t_2)}] \in \widehat{\alpha}(t_2) \\ [b^L_{(t_2)}, b^U_{(t_2)}] \in \widehat{\beta}(t_2) \\ [c^L_{(t_2)}, c^U_{(t_2)}] \in \widehat{\gamma}(t_2)}} \left\{ \left\{ \left[\frac{(\eta_{a^L_{(t_2)}})^{\omega(t_2)} - (\vartheta_{a^L_{(t_2)}})^{\omega(t_2)}}{(\eta_{a^L_{(t_2)}})^{\omega(t_2)} + (\vartheta_{a^L_{(t_2)}})^{\omega(t_2)}}, \frac{(\eta_{a^U_{(t_2)}})^{\omega(t_2)} - (\vartheta_{a^U_{(t_2)}})^{\omega(t_2)}}{(\eta_{a^U_{(t_2)}})^{\omega(t_2)} + (\vartheta_{a^U_{(t_2)}})^{\omega(t_2)}} \right] \right\}, \left[\frac{2(\theta_{b^L_{(t_2)}})^{\omega(t_2)}}{(\zeta_{b^L_{(t_2)}})^{\omega(t_2)} + (\theta_{b^L_{(t_2)}})^{\omega(t_2)}}, \frac{2(\theta_{b^U_{(t_2)}})^{\omega(t_2)}}{(\zeta_{b^U_{(t_2)}})^{\omega(t_2)} + (\theta_{b^U_{(t_2)}})^{\omega(t_2)}} \right], \left[\frac{2(\psi_{c^L_{(t_2)}})^{\omega(t_2)}}{(\xi_{c^L_{(t_2)}})^{\omega(t_2)} + (\psi_{c^L_{(t_2)}})^{\omega(t_2)}}, \frac{2(\psi_{c^U_{(t_2)}})^{\omega(t_2)}}{(\xi_{c^U_{(t_2)}})^{\omega(t_2)} + (\psi_{c^U_{(t_2)}})^{\omega(t_2)}} \right] \right\} .
$$

Then, we obtain that (from Definition 4.1 (a))

$$
\omega(t_1)(\widehat{\wp}(t_1))^\lambda \oplus_\epsilon \omega(t_2)(\widehat{\wp}(t_2))^\lambda
$$

$$
= \bigcup_{\substack{[a^L_{(t_k)}, a^U_{(t_k)}] \in \widehat{\alpha}(t_k) \ (k=1,2) \\ [b^L_{(t_k)}, b^U_{(t_k)}] \in \widehat{\beta}(t_k) \ (k=1,2) \\ [c^L_{(t_k)}, c^U_{(t_k)}] \in \widehat{\gamma}(t_k) \ (k=1,2)}} \left\{ \left\{ \left[\frac{(\eta_{a^L_{(t_1)}})^{\omega(t_1)}(\eta_{a^L_{(t_2)}})^{\omega(t_2)} - (\vartheta_{a^L_{(t_1)}})^{\omega(t_1)}(\vartheta_{a^L_{(t_2)}})^{\omega(t_2)}}{(\eta_{a^L_{(t_1)}})^{\omega(t_1)}(\eta_{a^L_{(t_2)}})^{\omega(t_2)} + (\vartheta_{a^L_{(t_1)}})^{\omega(t_1)}(\vartheta_{a^L_{(t_2)}})^{\omega(t_2)}}, \frac{(\eta_{a^U_{(t_1)}})^{\omega(t_1)}(\eta_{a^U_{(t_2)}})^{\omega(t_2)} - (\vartheta_{a^U_{(t_1)}})^{\omega(t_1)}(\vartheta_{a^U_{(t_2)}})^{\omega(t_2)}}{(\eta_{a^U_{(t_1)}})^{\omega(t_1)}(\eta_{a^U_{(t_2)}})^{\omega(t_2)} + (\vartheta_{a^U_{(t_1)}})^{\omega(t_1)}(\vartheta_{a^U_{(t_2)}})^{\omega(t_2)}} \right] \right\}, \left[\frac{2(\theta_{b^L_{(t_1)}})^{\omega(t_1)}(\theta_{b^L_{(t_2)}})^{\omega(t_2)}}{(\zeta_{b^L_{(t_1)}})^{\omega(t_1)}(\zeta_{b^L_{(t_2)}})^{\omega(t_2)} + (\theta_{b^L_{(t_1)}})^{\omega(t_1)}(\theta_{b^L_{(t_2)}})^{\omega(t_2)}}, \frac{2(\theta_{b^U_{(t_1)}})^{\omega(t_1)}(\theta_{b^U_{(t_2)}})^{\omega(t_2)}}{(\zeta_{b^U_{(t_1)}})^{\omega(t_1)}(\zeta_{b^U_{(t_2)}})^{\omega(t_2)} + (\theta_{b^U_{(t_1)}})^{\omega(t_1)}(\theta_{b^U_{(t_2)}})^{\omega(t_2)}} \right], \left[\frac{2(\psi_{c^L_{(t_1)}})^{\omega(t_1)}(\psi_{c^L_{(t_2)}})^{\omega(t_2)}}{(\xi_{c^L_{(t_1)}})^{\omega(t_1)}(\xi_{c^L_{(t_2)}})^{\omega(t_2)} + (\psi_{c^L_{(t_1)}})^{\omega(t_1)}(\psi_{c^L_{(t_2)}})^{\omega(t_2)}}, \frac{2(\psi_{c^U_{(t_1)}})^{\omega(t_1)}(\psi_{c^U_{(t_2)}})^{\omega(t_2)}}{(\xi_{c^U_{(t_1)}})^{\omega(t_1)}(\xi_{c^U_{(t_2)}})^{\omega(t_2)} + (\psi_{c^U_{(t_1)}})^{\omega(t_1)}(\psi_{c^U_{(t_2)}})^{\omega(t_2)}} \right] \right\}
$$

$$
= \bigcup_{\substack{[a^L_{(t_k)}, a^U_{(t_k)}] \in \widehat{\alpha}(t_k) \ (k=1,2) \\ [b^L_{(t_k)}, b^U_{(t_k)}] \in \widehat{\beta}(t_k) \ (k=1,2) \\ [c^L_{(t_k)}, c^U_{(t_k)}] \in \widehat{\gamma}(t_k) \ (k=1,2)}} \left\{ \left\{ \left[\frac{\prod_{k=1}^2 (\eta_{a^L_{(t_k)}})^{\omega(t_k)} - \prod_{k=1}^2 (\vartheta_{a^L_{(t_k)}})^{\omega(t_k)}}{\prod_{k=1}^2 (\eta_{a^L_{(t_k)}})^{\omega(t_k)} + \prod_{k=1}^2 (\vartheta_{a^L_{(t_k)}})^{\omega(t_k)}}, \frac{\prod_{k=1}^2 (\eta_{a^U_{(t_k)}})^{\omega(t_k)} - \prod_{k=1}^2 (\vartheta_{a^U_{(t_k)}})^{\omega(t_k)}}{\prod_{k=1}^2 (\eta_{a^U_{(t_k)}})^{\omega(t_k)} + \prod_{k=1}^2 (\vartheta_{a^U_{(t_k)}})^{\omega(t_k)}} \right] \right\}, \left[\frac{2 \prod_{k=1}^2 (\theta_{b^L_{(t_k)}})^{\omega(t_k)}}{\prod_{k=1}^2 (\zeta_{b^L_{(t_k)}})^{\omega(t_k)} + \prod_{k=1}^2 (\theta_{b^L_{(t_k)}})^{\omega(t_k)}}, \frac{2 \prod_{k=1}^2 (\theta_{b^U_{(t_k)}})^{\omega(t_k)}}{\prod_{k=1}^2 (\zeta_{b^U_{(t_k)}})^{\omega(t_k)} + \prod_{k=1}^2 (\theta_{b^U_{(t_k)}})^{\omega(t_k)}} \right], \left[\frac{2 \prod_{k=1}^2 (\psi_{c^L_{(t_k)}})^{\omega(t_k)}}{\prod_{k=1}^2 (\xi_{c^L_{(t_k)}})^{\omega(t_k)} + \prod_{k=1}^2 (\psi_{c^L_{(t_k)}})^{\omega(t_k)}}, \frac{2 \prod_{k=1}^2 (\psi_{c^U_{(t_k)}})^{\omega(t_k)}}{\prod_{k=1}^2 (\xi_{c^U_{(t_k)}})^{\omega(t_k)} + \prod_{k=1}^2 (\psi_{c^U_{(t_k)}})^{\omega(t_k)}} \right] \right\} \quad (4.15)
$$

and thus

$$
(\omega(t_1)(\widehat{\wp}(t_1))^\lambda \oplus_\epsilon \omega(t_2)(\widehat{\wp}(t_2))^\lambda)^{\frac{1}{\lambda}} = \bigcup_{\substack{[a^L_{(t_k)}, a^U_{(t_k)}] \in \widehat{\alpha}(t_k) \ (k=1,2) \\ [b^L_{(t_k)}, b^U_{(t_k)}] \in \widehat{\beta}(t_k) \ (k=1,2) \\ [c^L_{(t_k)}, c^U_{(t_k)}] \in \widehat{\gamma}(t_k) \ (k=1,2)}}
$$

$$\left\{\left\{\left[\frac{\left((\eta_{a_{(t_1)}^L})^{\omega(t_1)}(\eta_{a_{(t_2)}^L})^{\omega(t_2)}-(\vartheta_{a_{(t_1)}^L})^{\omega(t_1)}(\vartheta_{a_{(t_2)}^L})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}}{\left((\eta_{a_{(t_1)}^L})^{\omega(t_1)}(\eta_{a_{(t_2)}^L})^{\omega(t_2)}+3(\vartheta_{a_{(t_1)}^L})^{\omega(t_1)}(\vartheta_{a_{(t_2)}^L})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}+\left((\eta_{a_{(t_1)}^L})^{\omega(t_1)}(\eta_{a_{(t_2)}^L})^{\omega(t_2)}-(\vartheta_{a_{(t_1)}^L})^{\omega(t_1)}(\vartheta_{a_{(t_2)}^L})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}},\right.\right.$$

$$\left.\frac{\left((\eta_{a_{(t_1)}^U})^{\omega(t_1)}(\eta_{a_{(t_2)}^U})^{\omega(t_2)}-(\vartheta_{a_{(t_1)}^U})^{\omega(t_1)}(\vartheta_{a_{(t_2)}^U})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}}{\left((\eta_{a_{(t_1)}^U})^{\omega(t_1)}(\eta_{a_{(t_2)}^U})^{\omega(t_2)}+3(\vartheta_{a_{(t_1)}^U})^{\omega(t_1)}(\vartheta_{a_{(t_2)}^U})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}+\left((\eta_{a_{(t_1)}^U})^{\omega(t_1)}(\eta_{a_{(t_2)}^U})^{\omega(t_2)}-(\vartheta_{a_{(t_1)}^U})^{\omega(t_1)}(\vartheta_{a_{(t_2)}^U})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}}\right]$$

$$\left[\frac{\left((\zeta_{b_{(t_1)}^L})^{\omega(t_1)}(\zeta_{b_{(t_2)}^L})^{\omega(t_2)}+3(\theta_{b_{(t_1)}^L})^{\omega(t_1)}(\theta_{b_{(t_2)}^L})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}-\left((\zeta_{b_{(t_1)}^L})^{\omega(t_1)}(\zeta_{b_{(t_2)}^L})^{\omega(t_2)}-(\theta_{b_{(t_1)}^L})^{\omega(t_1)}(\theta_{b_{(t_2)}^L})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}}{\left((\zeta_{b_{(t_1)}^L})^{\omega(t_1)}(\zeta_{b_{(t_2)}^L})^{\omega(t_2)}+3(\theta_{b_{(t_1)}^L})^{\omega(t_1)}(\theta_{b_{(t_2)}^L})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}+\left((\zeta_{b_{(t_1)}^L})^{\omega(t_1)}(\zeta_{b_{(t_2)}^L})^{\omega(t_2)}-(\theta_{b_{(t_1)}^L})^{\omega(t_1)}(\theta_{b_{(t_2)}^L})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}},\right.$$

$$\left.\frac{\left((\zeta_{b_{(t_1)}^U})^{\omega(t_1)}(\zeta_{b_{(t_2)}^U})^{\omega(t_2)}+3(\theta_{b_{(t_1)}^U})^{\omega(t_1)}(\theta_{b_{(t_2)}^U})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}-\left((\zeta_{b_{(t_1)}^U})^{\omega(t_1)}(\zeta_{b_{(t_2)}^U})^{\omega(t_2)}-(\theta_{b_{(t_1)}^U})^{\omega(t_1)}(\theta_{b_{(t_2)}^U})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}}{\left((\zeta_{b_{(t_1)}^U})^{\omega(t_1)}(\zeta_{b_{(t_2)}^U})^{\omega(t_2)}+3(\theta_{b_{(t_1)}^U})^{\omega(t_1)}(\theta_{b_{(t_2)}^U})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}+\left((\zeta_{b_{(t_1)}^U})^{\omega(t_1)}(\zeta_{b_{(t_2)}^U})^{\omega(t_2)}-(\theta_{b_{(t_1)}^U})^{\omega(t_1)}(\theta_{b_{(t_2)}^U})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}}\right]$$

$$\left[\frac{\left((\xi_{c_{(t_1)}^L})^{\omega(t_1)}(\xi_{c_{(t_2)}^L})^{\omega(t_2)}+3(\psi_{c_{(t_1)}^L})^{\omega(t_1)}(\psi_{c_{(t_2)}^L})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}-\left((\xi_{c_{(t_1)}^L})^{\omega(t_1)}(\xi_{c_{(t_2)}^L})^{\omega(t_2)}-(\psi_{c_{(t_1)}^L})^{\omega(t_1)}(\psi_{c_{(t_2)}^L})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}}{\left((\xi_{c_{(t_1)}^L})^{\omega(t_1)}(\xi_{c_{(t_2)}^L})^{\omega(t_2)}+3(\psi_{c_{(t_1)}^L})^{\omega(t_1)}(\psi_{c_{(t_2)}^L})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}+\left((\xi_{c_{(t_1)}^L})^{\omega(t_1)}(\xi_{c_{(t_2)}^L})^{\omega(t_2)}-(\psi_{c_{(t_1)}^L})^{\omega(t_1)}(\psi_{c_{(t_2)}^L})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}},\right.$$

$$\left.\left.\left.\frac{\left((\xi_{c_{(t_1)}^U})^{\omega(t_1)}(\xi_{c_{(t_2)}^U})^{\omega(t_2)}+3(\psi_{c_{(t_1)}^U})^{\omega(t_1)}(\psi_{c_{(t_2)}^U})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}-\left((\xi_{c_{(t_1)}^U})^{\omega(t_1)}(\xi_{c_{(t_2)}^U})^{\omega(t_2)}-(\psi_{c_{(t_1)}^U})^{\omega(t_1)}(\psi_{c_{(t_2)}^U})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}}{\left((\xi_{c_{(t_1)}^U})^{\omega(t_1)}(\xi_{c_{(t_2)}^U})^{\omega(t_2)}+3(\psi_{c_{(t_1)}^U})^{\omega(t_1)}(\psi_{c_{(t_2)}^U})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}+\left((\xi_{c_{(t_1)}^U})^{\omega(t_1)}(\xi_{c_{(t_2)}^U})^{\omega(t_2)}-(\psi_{c_{(t_1)}^U})^{\omega(t_1)}(\psi_{c_{(t_2)}^U})^{\omega(t_2)}\right)^{\frac{1}{\lambda}}}\right]\right\}\right\}.$$

$$= GDIVPHFEWA_\epsilon^{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2)).$$

So, Eq. (4.11) is valid for $r = 2$.

- For $r = s$:
  If Eq. (4.11) is valid, we have the following

$$GDIVPHFEWA_\epsilon^{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_s))$$

$$= \bigcup_{\substack{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k)(k=1,2,...,s) \\ [b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k)(k=1,2,...,s) \\ [c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k)(k=1,2,...,s)}} \left\{\left\{\left[\frac{2\left(\prod_{k=1}^s(\eta_{a_{(t_k)}^L})^{\omega(t_k)}-\prod_{k=1}^s(\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^s(\eta_{a_{(t_k)}^L})^{\omega(t_k)}+3\prod_{k=1}^s(\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}+\left(\prod_{k=1}^s(\eta_{a_{(t_k)}^L})^{\omega(t_k)}-\prod_{k=1}^s(\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}},\right.\right.\right.$$

$$\left.\frac{2\left(\prod_{k=1}^s(\eta_{a_{(t_k)}^U})^{\omega(t_k)}-\prod_{k=1}^s(\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^s(\eta_{a_{(t_k)}^U})^{\omega(t_k)}+3\prod_{k=1}^s(\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}+\left(\prod_{k=1}^s(\eta_{a_{(t_k)}^U})^{\omega(t_k)}-\prod_{k=1}^s(\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}\right]$$

$$\left[\frac{\left(\prod_{k=1}^s(\zeta_{b_{(t_k)}^L})^{\omega(t_k)}+3\prod_{k=1}^s(\theta_{b_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}-\left(\prod_{k=1}^s(\zeta_{b_{(t_k)}^L})^{\omega(t_k)}-\prod_{k=1}^s(\theta_{b_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^s(\zeta_{b_{(t_k)}^L})^{\omega(t_k)}+3\prod_{k=1}^s(\theta_{b_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}+\left(\prod_{k=1}^s(\zeta_{b_{(t_k)}^L})^{\omega(t_k)}-\prod_{k=1}^s(\theta_{b_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}},\right.$$

$$\left.\frac{\left(\prod_{k=1}^s(\zeta_{b_{(t_k)}^U})^{\omega(t_k)}+3\prod_{k=1}^s(\theta_{b_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}-\left(\prod_{k=1}^s(\zeta_{b_{(t_k)}^U})^{\omega(t_k)}-\prod_{k=1}^s(\theta_{b_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^s(\zeta_{b_{(t_k)}^U})^{\omega(t_k)}+3\prod_{k=1}^s(\theta_{b_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}+\left(\prod_{k=1}^s(\zeta_{b_{(t_k)}^U})^{\omega(t_k)}-\prod_{k=1}^s(\theta_{b_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}\right]$$

$$\left[\frac{\left(\prod_{k=1}^s(\xi_{c_{(t_k)}^L})^{\omega(t_k)}+3\prod_{k=1}^s(\psi_{c_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}-\left(\prod_{k=1}^s(\xi_{c_{(t_k)}^L})^{\omega(t_k)}-\prod_{k=1}^s(\psi_{c_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^s(\xi_{c_{(t_k)}^L})^{\omega(t_k)}+3\prod_{k=1}^s(\psi_{c_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}+\left(\prod_{k=1}^s(\xi_{c_{(t_k)}^L})^{\omega(t_k)}-\prod_{k=1}^s(\psi_{c_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}},\right.$$

$$\left.\left.\left.\frac{\left(\prod_{k=1}^s(\xi_{c_{(t_k)}^U})^{\omega(t_k)}+3\prod_{k=1}^s(\psi_{c_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}-\left(\prod_{k=1}^s(\xi_{c_{(t_k)}^U})^{\omega(t_k)}-\prod_{k=1}^s(\psi_{c_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^s(\xi_{c_{(t_k)}^U})^{\omega(t_k)}+3\prod_{k=1}^s(\psi_{c_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}+\left(\prod_{k=1}^s(\xi_{c_{(t_k)}^U})^{\omega(t_k)}-\prod_{k=1}^s(\psi_{c_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}\right]\right\}\right\}$$

Now let's demonstrate that Eq. (4.11) holds when $r = s + 1$.
In Eq. (4.15), s is substituted for 2 and also by Eqs. (4.3), (4.4), we write

$$\omega(t_1)(\widehat{\wp}(t_1))^\lambda \oplus_\epsilon \omega(t_2)(\widehat{\wp}(t_2))^\lambda \oplus_\epsilon ... \oplus_\epsilon \omega(t_s)(\widehat{\wp}(t_s))^\lambda \oplus_\epsilon \omega(t_{s+1})(\widehat{\wp}(t_{s+1}))^\lambda$$

$$= \bigcup_{\substack{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k)\ (k=1,2,\dots,s) \\ [b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k)\ (k=1,2,\dots,s) \\ [c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k)\ (k=1,2,\dots,s)}} \left\{ \left\{ \begin{array}{l} \left[\frac{\prod_{k=1}^{s}(\eta_{a_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^{s}(\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}}{\prod_{k=1}^{s}(\eta_{a_{(t_k)}^L})^{\omega(t_k)} + \prod_{k=1}^{s}(\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}}, \frac{\prod_{k=1}^{s}(\eta_{a_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^{s}(\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}}{\prod_{k=1}^{s}(\eta_{a_{(t_k)}^U})^{\omega(t_k)} + \prod_{k=1}^{s}(\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}} \right] \\ \left[\frac{2\prod_{k=1}^{s}(\theta_{b_{(t_k)}^L})^{\omega(t_k)}}{\prod_{k=1}^{s}(\zeta_{b_{(t_k)}^L})^{\omega(t_k)} + \prod_{k=1}^{s}(\theta_{b_{(t_k)}^L})^{\omega(t_k)}}, \frac{2\prod_{k=1}^{s}(\theta_{b_{(t_k)}^U})^{\omega(t_k)}}{\prod_{k=1}^{s}(\zeta_{b_{(t_k)}^U})^{\omega(t_k)} + \prod_{k=1}^{s}(\theta_{b_{(t_k)}^U})^{\omega(t_k)}} \right] \\ \left[\frac{2\prod_{k=1}^{s}(\psi_{c_{(t_k)}^L})^{\omega(t_k)}}{\prod_{k=1}^{s}(\xi_{c_{(t_k)}^L})^{\omega(t_k)} + \prod_{k=1}^{s}(\psi_{c_{(t_k)}^L})^{\omega(t_k)}}, \frac{2\prod_{k=1}^{s}(\psi_{c_{(t_k)}^U})^{\omega(t_k)}}{\prod_{k=1}^{s}(\xi_{c_{(t_k)}^U})^{\omega(t_k)} + \prod_{k=1}^{s}(\psi_{c_{(t_k)}^U})^{\omega(t_k)}} \right] \end{array} \right\} \right\} \oplus_\epsilon$$

$$\bigcup_{\substack{[a_{(t_{s+1})}^L, a_{(t_{s+1})}^U] \in \widehat{\alpha}(t_{s+1}) \\ [b_{(t_{s+1})}^L, b_{(t_{s+1})}^U] \in \widehat{\beta}(t_{s+1}) \\ [c_{(t_{s+1})}^L, c_{(t_{s+1})}^U] \in \widehat{\gamma}(t_{s+1})}} \left\{ \left\{ \begin{array}{l} \left[\frac{(\eta_{a_{(t_{s+1})}^L})^{\omega(t_{s+1})} - (\vartheta_{a_{(t_{s+1})}^L})^{\omega(t_{s+1})}}{(\eta_{a_{(t_{s+1})}^L})^{\omega(t_{s+1})} + (\vartheta_{a_{(t_{s+1})}^L})^{\omega(t_{s+1})}}, \frac{(\eta_{a_{(t_{s+1})}^U})^{\omega(t_{s+1})} - (\vartheta_{a_{(t_{s+1})}^U})^{\omega(t_{s+1})}}{(\eta_{a_{(t_{s+1})}^U})^{\omega(t_{s+1})} + (\vartheta_{a_{(t_{s+1})}^U})^{\omega(t_{s+1})}} \right] \\ \left[\frac{2(\theta_{b_{(t_{s+1})}^L})^{\omega(t_{s+1})}}{(\zeta_{b_{(t_{s+1})}^L})^{\omega(t_{s+1})} + (\theta_{b_{(t_{s+1})}^L})^{\omega(t_{s+1})}}, \frac{2(\theta_{b_{(t_{s+1})}^U})^{\omega(t_{s+1})}}{(\zeta_{b_{(t_{s+1})}^U})^{\omega(t_{s+1})} + (\theta_{b_{(t_{s+1})}^U})^{\omega(t_{s+1})}} \right] \\ \left[\frac{2(\psi_{c_{(t_{s+1})}^L})^{\omega(t_{s+1})}}{(\xi_{c_{(t_{s+1})}^L})^{\omega(t_{s+1})} + (\psi_{c_{(t_{s+1})}^L})^{\omega(t_{s+1})}}, \frac{2(\psi_{c_{(t_{s+1})}^U})^{\omega(t_{s+1})}}{(\xi_{c_{(t_{s+1})}^U})^{\omega(t_{s+1})} + (\psi_{c_{(t_{s+1})}^U})^{\omega(t_{s+1})}} \right] \end{array} \right\} \right\}$$

$$= \bigcup_{\substack{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k)\ (k=1,2,\dots,s+1) \\ [b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k)\ (k=1,2,\dots,s+1) \\ [c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k)\ (k=1,2,\dots,s+1)}} \left\{ \left\{ \begin{array}{l} \left[\frac{\prod_{k=1}^{s+1}(\eta_{a_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^{s+1}(\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}}{\prod_{k=1}^{s+1}(\eta_{a_{(t_k)}^L})^{\omega(t_k)} + \prod_{k=1}^{s+1}(\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}}, \frac{\prod_{k=1}^{s+1}(\eta_{a_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^{s+1}(\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}}{\prod_{k=1}^{s+1}(\eta_{a_{(t_k)}^U})^{\omega(t_k)} + \prod_{k=1}^{s+1}(\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}} \right] \\ \left[\frac{2\prod_{k=1}^{s+1}(\theta_{b_{(t_k)}^L})^{\omega(t_k)}}{\prod_{k=1}^{s+1}(\zeta_{b_{(t_k)}^L})^{\omega(t_k)} + \prod_{k=1}^{s+1}(\theta_{b_{(t_k)}^L})^{\omega(t_k)}}, \frac{2\prod_{k=1}^{s+1}(\theta_{b_{(t_k)}^U})^{\omega(t_k)}}{\prod_{k=1}^{s+1}(\zeta_{b_{(t_k)}^U})^{\omega(t_k)} + \prod_{k=1}^{s+1}(\theta_{b_{(t_k)}^U})^{\omega(t_k)}} \right] \\ \left[\frac{2\prod_{k=1}^{s+1}(\psi_{c_{(t_k)}^L})^{\omega(t_k)}}{\prod_{k=1}^{s+1}(\xi_{c_{(t_k)}^L})^{\omega(t_k)} + \prod_{k=1}^{s+1}(\psi_{c_{(t_k)}^L})^{\omega(t_k)}}, \frac{2\prod_{k=1}^{s+1}(\psi_{c_{(t_k)}^U})^{\omega(t_k)}}{\prod_{k=1}^{s+1}(\xi_{c_{(t_k)}^U})^{\omega(t_k)} + \prod_{k=1}^{s+1}(\psi_{c_{(t_k)}^U})^{\omega(t_k)}} \right] \end{array} \right\} \right\}.$$

Then, we obtain that (from Definition 4.1 (d))

$$(\omega(t_1)(\widehat{\wp}(t_1))^\lambda \oplus_\epsilon \omega(t_2)(\widehat{\wp}(t_2))^\lambda \oplus_\epsilon \dots \oplus_\epsilon \omega(t_s)(\widehat{\wp}(t_s))^\lambda \oplus_\epsilon \omega(t_{s+1})(\widehat{\wp}(t_{s+1}))^\lambda)^{\frac{1}{\lambda}}$$

$$= \bigcup_{\substack{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k)(k=1,2,\dots,s+1) \\ [b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k)(k=1,2,\dots,s+1) \\ [c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k)(k=1,2,\dots,s+1)}} \left\{ \left\{ \begin{array}{l} \left[\frac{2\left(\prod_{k=1}^{s+1}(\eta_{a_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^{s+1}(\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{s+1}(\eta_{a_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^{s+1}(\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{s+1}(\eta_{a_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^{s+1}(\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}, \right. \\ \left. \frac{2\left(\prod_{k=1}^{s+1}(\eta_{a_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^{s+1}(\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{s+1}(\eta_{a_{(t_k)}^U})^{\omega(t_k)} + 3\prod_{k=1}^{s+1}(\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{s+1}(\eta_{a_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^{s+1}(\vartheta_{a_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}} \right] \\ \left[\frac{\left(\prod_{k=1}^{s+1}(\zeta_{b_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^{s+1}(\theta_{b_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{s+1}(\zeta_{b_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^{s+1}(\theta_{b_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{s+1}(\zeta_{b_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^{s+1}(\theta_{b_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{s+1}(\zeta_{b_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^{s+1}(\theta_{b_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}, \right. \\ \left. \frac{\left(\prod_{k=1}^{s+1}(\zeta_{b_{(t_k)}^U})^{\omega(t_k)} + 3\prod_{k=1}^{s+1}(\theta_{b_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{s+1}(\zeta_{b_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^{s+1}(\theta_{b_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{s+1}(\zeta_{b_{(t_k)}^U})^{\omega(t_k)} + 3\prod_{k=1}^{s+1}(\theta_{b_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{s+1}(\zeta_{b_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^{s+1}(\theta_{b_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}} \right] \\ \left[\frac{\left(\prod_{k=1}^{s+1}(\xi_{c_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^{s+1}(\psi_{c_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{s+1}(\xi_{c_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^{s+1}(\psi_{c_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{s+1}(\xi_{c_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^{s+1}(\psi_{c_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{s+1}(\xi_{c_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^{s+1}(\psi_{c_{(t_k)}^L})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}, \right. \\ \left. \frac{\left(\prod_{k=1}^{s+1}(\xi_{c_{(t_k)}^U})^{\omega(t_k)} + 3\prod_{k=1}^{s+1}(\psi_{c_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{s+1}(\xi_{c_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^{s+1}(\psi_{c_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{s+1}(\xi_{c_{(t_k)}^U})^{\omega(t_k)} + 3\prod_{k=1}^{s+1}(\psi_{c_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{s+1}(\xi_{c_{(t_k)}^U})^{\omega(t_k)} - \prod_{k=1}^{s+1}(\psi_{c_{(t_k)}^U})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}} \right] \end{array} \right\} \right\}$$

$$= GDIVPHFEWA_\epsilon^{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), \dots, \widehat{\wp}(t_{s+1}))$$

Thus, Eq. (4.11) is true for $r = s + 1$.

Consequently, we say that Eq. (4.11) holds for all values of r.   □

Note 3. If $\lambda = 1$ then GDIVPHFEWA and GDIVPHFEWG operators reduce to DIVPH-FEWA and DIVPHFEWG operators, respectively. Therefore, we will investigate the results of GDIPHFWA and GDIVPHFWG operators.

Theorem 4.6 (Idempotency) *Let $\widehat{\wp}(t_k)$ ($k = 1, 2, ..., r$) be the IVPHFEs collected from k different periods t_k ($k = 1, 2, ..., r$). If $\widehat{\wp}(t_k) = \widehat{\wp} = \{\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}\}$ for all $k = 1, 2, ..., r$ then*

1. $GDIVPHFEWA_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) = \widehat{\wp}.$
2. $GDIVPHFEWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_2)) = \widehat{\wp}.$

Proof Let us prove $GDIVPHFEWA_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) = \widehat{\wp}$, the other can be demonstrated similar to this.

Considering $\widehat{\wp}(t_k) = \widehat{\wp} = \{\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}\}$, by Theorem 4.5 (3) we have

$GDIVPHFEWA_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r))$

$$= ((\lambda(\widehat{\wp}(t_1))^{\omega(t_1)} \oplus_\epsilon \lambda(\widehat{\wp}(t_2))^{\omega(t_2)} \oplus_\epsilon ... \oplus_\epsilon \lambda(\widehat{\wp}(t_r))^{\omega(t_r)}))^{\frac{1}{\lambda}}$$

$$= \bigcup_{[a^L,a^U]\in\widehat{\alpha},[b^L,b^U]\in\widehat{\beta},[c^L,c^U]\in\widehat{\gamma}} \left\{ \left\{ \begin{bmatrix} \frac{2\left(\prod_{k=1}^r(\eta_{aL})^{\omega(t_k)}-\prod_{k=1}^r(\vartheta_{aL})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r(\eta_{aL})^{\omega(t_k)}+3\prod_{k=1}^r(\vartheta_{aL})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}+\left(\prod_{k=1}^r(\eta_{aL})^{\omega(t_k)}-\prod_{k=1}^r(\vartheta_{aL})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}, \\ \frac{2\left(\prod_{k=1}^r(\eta_{aU})^{\omega(t_k)}-\prod_{k=1}^r(\vartheta_{aU})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r(\eta_{aU})^{\omega(t_k)}+3\prod_{k=1}^r(\vartheta_{aU})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}+\left(\prod_{k=1}^r(\eta_{aU})^{\omega(t_k)}-\prod_{k=1}^r(\vartheta_{aU})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}} \end{bmatrix} \right\} \right.$$

$$\left. \begin{bmatrix} \frac{\left(\prod_{k=1}^r(\zeta_{bL})^{\omega(t_k)}+3\prod_{k=1}^r(\theta_{bL})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}-\left(\prod_{k=1}^r(\zeta_{bL})^{\omega(t_k)}-\prod_{k=1}^r(\theta_{bL})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r(\zeta_{bL})^{\omega(t_k)}+3\prod_{k=1}^r(\theta_{bL})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}+\left(\prod_{k=1}^r(\zeta_{bL})^{\omega(t_k)}-\prod_{k=1}^r(\theta_{bL})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}, \\ \frac{\left(\prod_{k=1}^r(\zeta_{bU})^{\omega(t_k)}+3\prod_{k=1}^r(\theta_{bU})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}-\left(\prod_{k=1}^r(\zeta_{bU})^{\omega(t_k)}-\prod_{k=1}^r(\theta_{bU})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r(\zeta_{bU})^{\omega(t_k)}+3\prod_{k=1}^r(\theta_{bU})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}+\left(\prod_{k=1}^r(\zeta_{bU})^{\omega(t_k)}-\prod_{k=1}^r(\theta_{bU})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}} \end{bmatrix} \right.$$

$$\left. \begin{bmatrix} \frac{\left(\prod_{k=1}^r(\xi_{cL})^{\omega(t_k)}+3\prod_{k=1}^r(\psi_{cL})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}-\left(\prod_{k=1}^r(\xi_{cL})^{\omega(t_k)}-\prod_{k=1}^r(\psi_{cL})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r(\xi_{cL})^{\omega(t_k)}+3\prod_{k=1}^r(\psi_{cL})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}+\left(\prod_{k=1}^r(\xi_{cL})^{\omega(t_k)}-\prod_{k=1}^r(\psi_{cL})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}, \\ \frac{\left(\prod_{k=1}^r(\xi_{cU})^{\omega(t_k)}+3\prod_{k=1}^r(\psi_{cU})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}-\left(\prod_{k=1}^r(\xi_{cU})^{\omega(t_k)}-\prod_{k=1}^r(\psi_{cU})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r(\xi_{cU})^{\omega(t_k)}+3\prod_{k=1}^r(\psi_{cU})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}+\left(\prod_{k=1}^r(\xi_{cU})^{\omega(t_k)}-\prod_{k=1}^r(\psi_{cU})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}} \end{bmatrix} \right\}$$

$$= \bigcup_{[a^L,a^U]\in\widehat{\alpha},[b^L,b^U]\in\widehat{\beta},[c^L,c^U]\in\widehat{\gamma}} \left\{ \left\{ \begin{bmatrix} \frac{2(\eta_{aL}-\vartheta_{aL})^{\frac{1}{\lambda}}}{(\eta_{aL}+3\vartheta_{aL})^{\frac{1}{\lambda}}+(\eta_{aL}-\vartheta_{aL})^{\frac{1}{\lambda}}}, & \frac{2(\eta_{aU}-\vartheta_{aU})^{\frac{1}{\lambda}}}{(\eta_{aU}+3\vartheta_{aU})^{\frac{1}{\lambda}}+(\eta_{aU}-\vartheta_{aU})^{\frac{1}{\lambda}}} \end{bmatrix} \\ \begin{bmatrix} \frac{(\zeta_{bL}+3\theta_{bL})^{\frac{1}{\lambda}}-(\zeta_{bL}-\theta_{bL})^{\frac{1}{\lambda}}}{(\zeta_{bL}+3\theta_{bL})^{\frac{1}{\lambda}}+(\zeta_{bL}-\theta_{bL})^{\frac{1}{\lambda}}}, & \frac{(\zeta_{bU}+3\theta_{bU})^{\frac{1}{\lambda}}-(\zeta_{bU}-\theta_{bU})^{\frac{1}{\lambda}}}{(\zeta_{bU}+3\theta_{bU})^{\frac{1}{\lambda}}+(\zeta_{bU}-\theta_{bU})^{\frac{1}{\lambda}}} \end{bmatrix} \\ \begin{bmatrix} \frac{(\xi_{cL}+3\psi_{cL})^{\frac{1}{\lambda}}-(\xi_{cL}-\psi_{cL})^{\frac{1}{\lambda}}}{(\xi_{cL}+3\psi_{cL})^{\frac{1}{\lambda}}+(\xi_{cL}-\psi_{cL})^{\frac{1}{\lambda}}}, & \frac{(\xi_{cU}+3\psi_{cU})^{\frac{1}{\lambda}}-(\xi_{cU}-\psi_{cU})^{\frac{1}{\lambda}}}{(\xi_{cU}+3\psi_{cU})^{\frac{1}{\lambda}}+(\xi_{cU}-\psi_{cU})^{\frac{1}{\lambda}}} \end{bmatrix} \end{bmatrix} \right\} \right\}$$

$$= \bigcup_{[a^L,a^U]\in\widehat{\alpha},[b^L,b^U]\in\widehat{\beta},[c^L,c^U]\in\widehat{\gamma}} \left\{ \begin{matrix} \{[a^L, a^U]\} \\ \{[b^L, b^U]\} \\ \{[c^L, c^U]\} \end{matrix} \right\} \text{ (by considering Eq. (4.12))}$$

Hence, the proof is completed. $\square$

Theorem 4.7 *(Boundedness) Let $\widehat{\wp}(t_k)$ ($k = 1, 2, ..., r$) be the IVPHFEs collected from k different periods t_k ($k = 1, 2, ..., r$).*

1. $\widehat{\wp}_{\min} \leq GDIVPHFEWA_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) \leq \widehat{\wp}_{\max}$
2. $\widehat{\wp}_{\min} \leq GDIVPHFEWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) \leq \widehat{\wp}_{\max}$

where

$$\widehat{\wp}_{\min} = \{\widehat{\alpha}_{\min}, \widehat{\beta}_{\max}, \widehat{\gamma}_{\max}\} = \left\{ \left\{ \left[a_{\min}^L, a_{\min}^U \right] \right\}, \left\{ \left[b_{\max}^L, b_{\max}^U \right] \right\}, \left\{ \left[c_{\max}^L, c_{\max}^U \right] \right\} \right\},$$

$$\widehat{\wp}_{\max} = \{\widehat{\alpha}_{\max}, \widehat{\beta}_{\min}, \widehat{\gamma}_{\min}\} = \left\{ \left\{ \left[a_{\max}^L, a_{\max}^U \right] \right\}, \left\{ \left[b_{\min}^L, b_{\min}^U \right] \right\}, \left\{ \left[c_{\min}^L, c_{\min}^U \right] \right\} \right\}$$

for

$$a_{\min}^L = \bigcup_{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k)} \min\{a_{(t_k)}^L\}, \quad a_{\min}^U = \bigcup_{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k)} \min\{a_{(t_k)}^U\},$$

$$a_{\max}^L = \bigcup_{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k)} max\{a_{(t_k)}^L\},$$

$$a_{\max}^U = \bigcup_{[a_{(t_k)}^L, a_{(t_k)}^U] \in \widehat{\alpha}(t_k)} \max\{a^U\}, \quad b_{\min}^L = \bigcup_{[b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k)} \min\{b_{(t_k)}^L\},$$

$$b_{\min}^U = \bigcup_{[b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k)} \min\{b_{(t_k)}^U\},$$

$$b_{\max}^L = \bigcup_{[b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k)} \max\{b_{(t_k)}^L\}, \quad b_{\max}^U = \bigcup_{[b_{(t_k)}^L, b_{(t_k)}^U] \in \widehat{\beta}(t_k)} \max\{b_{(t_k)}^U\},$$

$$c_{\min}^L = \bigcup_{[c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k)} min\{c_{(t_k)}^L\},$$

$$c_{\min}^U = \bigcup_{[c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k)} \min\{c_{(t_k)}^U\}, \quad c_{\max}^L = \bigcup_{[c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k)} \max\{c_{(t_k)}^L\},$$

$$c_{\max}^U = \bigcup_{[c_{(t_k)}^L, c_{(t_k)}^U] \in \widehat{\gamma}(t_k)} \max\{c_{(t_k)}^U\}.$$

Proof By considering Eqs. (4.11) and (4.13), they can be interpreted similar to the proof of Theorem 3.8. □

Theorem 4.8 (*Monotonicity*) *Let* $(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r))$ *and* $(\widehat{\wp}^*(t_1), \widehat{\wp}^*(t_2), ..., \widehat{\wp}^*(t_r))$ *be two collections of the IVPHFEs collected from k different periods* t_k ($k = 1, 2, ..., r$). *If* $\widehat{\wp}(t_k) \leq \widehat{\wp}^*(t_k)$ *for all* $k = 1, 2, ..., r$ *then*

1. $GDIVPHFEWA_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) \leq GDIVPHFEWA_{\omega(t)}(\widehat{\wp}^*(t_1), \widehat{\wp}^*(t_2), ..., \widehat{\wp}^*(t_r))$.
2. $GDIVPHFEWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) \leq GDIVPHFEWG_{\omega(t)}(\widehat{\wp}^*(t_1), \widehat{\wp}^*(t_2), ..., \widehat{\wp}^*(t_r))$.

Proof 1. The function $f(y) = \frac{(2-y)^\lambda - y^\lambda}{(2-y)^\lambda + 3y^\lambda}$ is the decreasing function, where $y \in [0, 1]$ and $\lambda > 0$. Since $\widehat{\wp}(t_k) \leq \widehat{\wp}^*(t_k)$ for all $k = 1, 2, ..., r$, we have $0 \leq a_{(t_k)}^L \leq a_{(t_k)}^{L^*}$ for all $k = 1, 2, ..., r$. Then, $f(a_{(t_k)}^L) \geq f(a_{(t_k)}^{L^*})$ for all $k = 1, 2, ..., r$, i. e., $\frac{(2-a_{(t_k)}^L)^\lambda - (a_{(t_k)}^L)^\lambda}{(2-a_{(t_k)}^L)^\lambda + 3(a_{(t_k)}^L)^\lambda} \geq$ $\frac{(2-a_{(t_k)}^{L^*})^\lambda - (a_{(t_k)}^{L^*})^\lambda}{(2-a_{(t_k)}^{L^*})^\lambda + 3(a_{(t_k)}^{L^*})^\lambda} \Leftrightarrow \frac{\vartheta_{a_{(t_k)}^L}}{\eta_{a_{(t_k)}^L}} \geq \frac{\vartheta_{a_{(t_k)}^{L^*}}}{\eta_{a_{(t_k)}^{L^*}}}$ (where η and ϑ are notations given in Eq. (4.12)).

Let $\omega(t) = (\omega(t_1), \omega(t_2), ..., \omega(t_r))^T$ be the weighted vector of t_k ($k = 1, 2, ..., r$), with $\omega(t_k) \in [0, 1]$ and $\sum_{k=1}^r \omega(t_k) = 1$. Then, we have

$$\left(\frac{\vartheta_{a_{(t_k)}^L}}{\eta_{a_{(t_k)}^L}}\right)^{\omega(t_k)} \geq \left(\frac{\vartheta_{a_{(t_k)}^{L^*}}}{\eta_{a_{(t_k)}^{L^*}}}\right)^{\omega(t_k)} \Leftrightarrow \prod_{k=1}^r \left(\frac{\vartheta_{a_{(t_k)}^L}}{\eta_{a_{(t_k)}^L}}\right)^{\omega(t_k)} \geq \prod_{k=1}^r \left(\frac{\vartheta_{a_{(t_k)}^{L^*}}}{\eta_{a_{(t_k)}^{L^*}}}\right)^{\omega(t_k)} \Leftrightarrow 1 + 3\prod_{k=1}^r (\frac{\vartheta_{a_{(t_k)}^L}}{\eta_{a_{(t_k)}^L}})^{\omega(t_k)}$$

$$\geq 1 + 3\prod_{k=1}^r (\frac{\vartheta_{a_{(t_k)}^{L^*}}}{\eta_{a_{(t_k)}^{L^*}}})^{\omega(t_k)}$$

$$\Leftrightarrow \frac{\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^r (\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}}{\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)}} \geq \frac{\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} + 3\prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)}}{\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)}} \tag{4.16}$$

$$\times \left(\text{since } 1 - \prod_{k=1}^r (\frac{\vartheta_{a_{(t_k)}^L}}{\eta_{a_{(t_k)}^L}})^{\omega(t_k)} \geq 1 - \prod_{k=1}^r (\frac{\vartheta_{a_{(t_k)}^{L*}}}{\eta_{a_{(t_k)}^{L*}}})^{\omega(t_k)} \Leftrightarrow \frac{\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)}}{\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^r (\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}} \geq \frac{\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)}}{\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} - \prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)}} \right)$$

$$\Leftrightarrow \frac{\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^r (\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}}{\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^r (\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}} \geq \frac{\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} + 3\prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)}}{\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} - \prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)}} \tag{4.17}$$

$$\Leftrightarrow \left(\frac{\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^r (\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}}{\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^r (\vartheta_{a_{(t_k)}^L})^{\omega(t_k)}} \right)^{\frac{1}{\lambda}} + 1 \geq \left(\frac{\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} + 3\prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)}}{\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} - \prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)}} \right)^{\frac{1}{\lambda}} + 1 \tag{4.18}$$

$$\Leftrightarrow \frac{\left(\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^r (\vartheta_{a_{(t_k)}^L})^{\omega(t_k)} \right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^r (\vartheta_{a_{(t_k)}^L})^{\omega(t_k)} \right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^r (\vartheta_{a_{(t_k)}^L})^{\omega(t_k)} \right)^{\frac{1}{\lambda}}}$$
$$\leq \frac{\left(\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} - \prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} \right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} + 3\prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} \right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} - \prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} \right)^{\frac{1}{\lambda}}} \tag{4.19}$$

$$\Leftrightarrow \frac{2\left(\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^r (\vartheta_{a_{(t_k)}^L})^{\omega(t_k)} \right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)} + 3\prod_{k=1}^r (\vartheta_{a_{(t_k)}^L})^{\omega(t_k)} \right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^r (\eta_{a_{(t_k)}^L})^{\omega(t_k)} - \prod_{k=1}^r (\vartheta_{a_{(t_k)}^L})^{\omega(t_k)} \right)^{\frac{1}{\lambda}}}$$
$$\leq \frac{2\left(\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} - \prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} \right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} + 3\prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} \right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} - \prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^{L*}}\right)^{\omega(t_k)} \right)^{\frac{1}{\lambda}}}. \tag{4.20}$$

Likewise, it is obtained that

$$\Leftrightarrow \frac{2\left(\prod_{k=1}^r \left(\eta_{a_{(t_k)}^U}\right)^{\omega(t_k)} - \prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^U}\right)^{\omega(t_k)} \right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r \left(\eta_{a_{(t_k)}^U}\right)^{\omega(t_k)} + 3\prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^U}\right)^{\omega(t_k)} \right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^r \left(\eta_{a_{(t_k)}^U}\right)^{\omega(t_k)} - \prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^U}\right)^{\omega(t_k)} \right)^{\frac{1}{\lambda}}}$$
$$\leq \frac{2\left(\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{U*}}\right)^{\omega(t_k)} - \prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^{U*}}\right)^{\omega(t_k)} \right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{U*}}\right)^{\omega(t_k)} + 3\prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^{U*}}\right)^{\omega(t_k)} \right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^r \left(\eta_{a_{(t_k)}^{U*}}\right)^{\omega(t_k)} - \prod_{k=1}^r \left(\vartheta_{a_{(t_k)}^{U*}}\right)^{\omega(t_k)} \right)^{\frac{1}{\lambda}}}. \tag{4.21}$$

The function $g(z) = \frac{(1+z)^\lambda + 3(1-z)^\lambda}{(1+z)^\lambda - (1-z)^\lambda}$ is the decreasing function, where $z \in (0, 1]$ and $\lambda > 0$. Since $\widehat{\wp}(t_k) \leq \widehat{\wp}^*(t_k)$ for all $k = 1, 2, ..., r$, we have $b_{(t_k)}^L \geq b_{(t_k)}^{L*} > 0$ for all $k =$

$1, 2, ..., r$. Then, $g(b^L_{(t_k)}) \leq g(b^{L^*}_{(t_k)})$ for all $k = 1, 2, ..., r$, i. e., $\frac{(1+b^L_{(t_k)})^\lambda + 3(1-b^L_{(t_k)})^\lambda}{(1+b^L_{(t_k)})^\lambda - (1-b^L_{(t_k)})^\lambda} \leq$

$\frac{(1+b^{L^*}_{(t_k)})^\lambda + 3(1-b^{L^*}_{(t_k)})^\lambda}{(1+b^{L^*}_{(t_k)})^\lambda - (1-b^{L^*}_{(t_k)})^\lambda} \Leftrightarrow \frac{\zeta_{b^L_{(t_k)}}}{\theta_{b^L_{(t_k)}}} \leq \frac{\zeta_{b^{L^*}_{(t_k)}}}{\theta_{b^{L^*}_{(t_k)}}}$ (where ζ and θ are notations given in Eq. (4.12)).

Let $\omega(t) = (\omega(t_1), \omega(t_2), ..., \omega(t_r))^T$ be the weighted vector of t_k ($k = 1, 2, ..., r$), with $\omega(t_k) \in [0, 1]$ and $\sum_{k=1}^{r} \omega(t_k) = 1$. Then, we have

$$\left(\frac{\zeta_{b^L_{(t_k)}}}{\theta_{b^L_{(t_k)}}}\right)^{\omega(t_k)} \leq \left(\frac{\zeta_{b^{L^*}_{(t_k)}}}{\theta_{b^L_{(t_k)}}}\right)^{\omega(t_k)} \Leftrightarrow \left(\frac{\theta_{b^L_{(t_k)}}}{\zeta_{b^L_{(t_k)}}}\right)^{\omega(t_k)} \geq \left(\frac{\theta_{b^{L^*}_{(t_k)}}}{\zeta_{b^{L^*}_{(t_k)}}}\right)^{\omega(t_k)} \Leftrightarrow 1 + 3 \prod_{k=1}^{r} \left(\frac{\theta_{b^L_{(t_k)}}}{\zeta_{b^L_{(t_k)}}}\right)^{\omega(t_k)}$$

$$\geq 1 + 3 \prod_{k=1}^{r} \left(\frac{\theta_{b^{L^*}_{(t_k)}}}{\zeta_{b^{L^*}_{(t_k)}}}\right)^{\omega(t_k)}$$

By proceeding similar to the calculations in (4.16), (4.17), (4.18), (4.19),

$$\Leftrightarrow \frac{2\left(\prod_{k=1}^{r}\left(\zeta_{a^L_{(t_k)}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\theta_{a^L_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}\left(\zeta_{a^L_{(t_k)}}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\theta_{a^L_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}\left(\zeta_{a^L_{(t_k)}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\theta_{a^L_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}$$

$$\leq \frac{2\left(\prod_{k=1}^{r}\left(\zeta_{a^{L^*}_{(t_k)}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\theta_{a^{L^*}_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}\left(\zeta_{a^{L^*}_{(t_k)}}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\theta_{a^{L^*}_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}\left(\zeta_{a^{L^*}_{(t_k)}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\theta_{a^{L^*}_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}} \quad (4.22)$$

$$\Leftrightarrow 1 - \frac{2\left(\prod_{k=1}^{r}\left(\zeta_{a^L_{(t_k)}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\theta_{a^L_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}\left(\zeta_{a^L_{(t_k)}}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\theta_{a^L_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}\left(\zeta_{a^L_{(t_k)}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\theta_{a^L_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}$$

$$\geq 1 - \frac{2\left(\prod_{k=1}^{r}\left(\zeta_{a^{L^*}_{(t_k)}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\theta_{a^{L^*}_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}\left(\zeta_{a^{L^*}_{(t_k)}}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\theta_{a^{L^*}_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}\left(\zeta_{a^{L^*}_{(t_k)}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\theta_{a^{L^*}_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}} \quad (4.23)$$

$$\Leftrightarrow \frac{\left(\prod_{k=1}^{r}\left(\zeta_{b^L_{(t_k)}}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\theta_{b^L_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{r}\left(\zeta_{b^L_{(t_k)}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\theta_{b^L_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}\left(\zeta_{b^L_{(t_k)}}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\theta_{b^L_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}\left(\zeta_{b^L_{(t_k)}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\theta_{b^L_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}$$

$$\geq \frac{\left(\prod_{k=1}^{r}\left(\zeta_{b^{L^*}_{(t_k)}}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\theta_{b^{L^*}_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{r}\left(\zeta_{b^{L^*}_{(t_k)}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\theta_{b^{L^*}_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}\left(\zeta_{b^{L^*}_{(t_k)}}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\theta_{b^{L^*}_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}\left(\zeta_{b^{L^*}_{(t_k)}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\theta_{b^{L^*}_{(t_k)}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}. \quad (4.24)$$

Note that Eq. (4.24) is valid even if $b^L_{(t_k)} = b^{L^*}_{(t_k)} = 0$ for all $k = 1, 2, ..., r$.

Likewise, it is obtained that

$$
\Leftrightarrow \frac{\left(\prod_{k=1}^{r}\left(\zeta_{b_{(t_k)}^U}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\theta_{b_{(t_k)}^U}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{r}\left(\zeta_{b_{(t_k)}^U}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\theta_{b_{(t_k)}^U}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}\left(\zeta_{b_{(t_k)}^U}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\theta_{b_{(t_k)}^U}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}\left(\zeta_{b_{(t_k)}^U}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\theta_{b_{(t_k)}^U}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}
$$

$$
\geq \frac{\left(\prod_{k=1}^{r}\left(\zeta_{b_{(t_k)}^{U*}}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\theta_{b_{(t_k)}^{U*}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{r}\left(\zeta_{b_{(t_k)}^{U*}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\theta_{b_{(t_k)}^{U*}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}\left(\zeta_{b_{(t_k)}^{U*}}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\theta_{b_{(t_k)}^{U*}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}\left(\zeta_{b_{(t_k)}^{U*}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\theta_{b_{(t_k)}^{U*}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}. \quad (4.25)
$$

Similarly, by considering the above function g, we have that

$$
\Leftrightarrow \frac{\left(\prod_{k=1}^{r}\left(\xi_{c_{(t_k)}^L}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\psi_{c_{(t_k)}^L}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{r}\left(\xi_{c_{(t_k)}^L}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\psi_{c_{(t_k)}^L}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}\left(\xi_{c_{(t_k)}^L}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\psi_{c_{(t_k)}^L}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}\left(\xi_{c_{(t_k)}^L}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\psi_{c_{(t_k)}^L}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}
$$

$$
\geq \frac{\left(\prod_{k=1}^{r}(\xi_{c_{(t_k)}^{L*}})^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\psi_{c_{(t_k)}^{L*}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{r}\left(\xi_{c_{(t_k)}^{L*}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\psi_{c_{(t_k)}^{L*}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}\left(\xi_{c_{(t_k)}^{L*}}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\psi_{c_{(t_k)}^{L*}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}\left(\xi_{c_{(t_k)}^{L*}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\psi_{c_{(t_k)}^{L*}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}} \quad (4.26)
$$

and

$$
\Leftrightarrow \frac{\left(\prod_{k=1}^{r}\left(\xi_{c_{(t_k)}^U}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\psi_{c_{(t_k)}^U}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{r}\left(\xi_{c_{(t_k)}^U}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\psi_{c_{(t_k)}^U}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}\left(\xi_{c_{(t_k)}^U}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\psi_{c_{(t_k)}^U}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}\left(\xi_{c_{(t_k)}^U}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\psi_{c_{(t_k)}^U}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}
$$

$$
\geq \frac{\left(\prod_{k=1}^{r}\left(\xi_{c_{(t_k)}^{U*}}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\psi_{c_{(t_k)}^{U*}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{r}\left(\xi_{c_{(t_k)}^{U*}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\psi_{c_{(t_k)}^{U*}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}\left(\xi_{c_{(t_k)}^{U*}}\right)^{\omega(t_k)} + 3\prod_{k=1}^{r}\left(\psi_{c_{(t_k)}^{U*}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}\left(\xi_{c_{(t_k)}^{U*}}\right)^{\omega(t_k)} - \prod_{k=1}^{r}\left(\psi_{c_{(t_k)}^{U*}}\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}. \quad (4.27)
$$

Thus, the proof of (1) is completed.

2. It can be shown similar to (1), therefore, omitted. $\qquad\square$

Theorem 4.9 *(Commutativity) Let $\widehat{\wp}(t_1)$, $\widehat{\wp}(t_2)$, ..., $\widehat{\wp}(t_r)$ be a collection of the IVPHFEs collected from k different periods t_k ($k = 1, 2, ..., r$). In addition, let $\widehat{\wp_n}(t_{1_n})$, $\widehat{\wp_n}(t_{2_n})$, ..., $\widehat{\wp_n}(t_{r_n})$ be a new permutation of $\widehat{\wp}(t_1)$, $\widehat{\wp}(t_2)$, ..., $\widehat{\wp}(t_r)$ then*

1. $GDIVPHFEWA_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) = GDIVPHFEWA_{\omega(t)}(\widehat{\wp_n}(t_{1_n}),$ $\widehat{\wp_n}(t_{2_n}), ..., \widehat{\wp_n}(t_{r_n}))$.
2. $GDIVPHFEWG_{\omega(t)}(\widehat{\wp}(t_1), \widehat{\wp}(t_2), ..., \widehat{\wp}(t_r)) = GDIVPHFEWG_{\omega(t)}(\widehat{\wp_n}(t_{1_n}),$ $\widehat{\wp_n}(t_{2_n}), ..., \widehat{\wp_n}(t_{r_n}))$.

Proof It can be easily seen by Theorem 4.5 (3) and (4). $\qquad\square$

5 The approaches to multi-period decision making under IVPHF environments

The MCDM is an important part of modern decision science. It has been extensively applied to various areas such as economics, society, management, military, etc., and has been receiving more and more attention in recent years. To date, however, most research has focused on single-period MCDM in which all the original decision information is collected at the same period and several methods have been proposed to solve this kind of problem. However, some complex DM problems have to consider the current and past performance of alternatives. This kind of DMs is called multi-period decision making (MPDM) and the original decision information is usually collected at different periods or refers to different moments in time. When the decision-making information takes form as interval-valued picture hesitant fuzzy information and is collected at different periods, the dynamic interval-valued picture hesitant fuzzy (Einstein) aggregation operators are suitable for dealing with such situations. In this part, we develop two approaches, using the proposed dynamic interval-valued picture hesitant fuzzy aggregation operators and dynamic interval-valued picture hesitant fuzzy Einstein aggregation operators, to multi-period multi-criteria decision making under the interval-valued picture hesitant fuzzy information.

The following notations are utilized to depict the possible numerical problems:

Let $Q = \{q_1, q_2, ..., q_n\}$ be a set of alternatives, and $L = \{\ell_1, \ell_2, ..., \ell_m\}$ be a set of parameters (criteria) and its weight vector be $\omega = (\omega_1, \omega_2, ..., \omega_m)^T$, where $\omega_j \in [0, 1]$ $(j = 1, 2, ..., m)$, $\sum_{j=1}^{m} \omega_j = 1$. In addition, $\mathcal{T} = \{t_1, t_2, ..., t_r\}$ be a set of time variables (or periods) and its weight vector be $\omega(t) = (\omega(t_1), \omega(t_2), ..., \omega(t_r))^T$ where $\omega(t_k) \in [0, 1]$, $\sum_{k=1}^{r} \omega(t_k) = 1$.

Assume that the IVPHF decision matrix created by the expert (decision maker) at the periods t_k $(k = 1, 2, ..., r)$ to measure the degrees of the alternative q_i $(i = 1, 2, ..., n)$ satisfies the parameter ℓ_j $(j = 1, 2, ..., m)$ is $D(t_k) = (d_{ij}(t_k))_{n \times m}$ where

$$d_{ij}(t_k) = \bigcup_{\substack{[a^L_{ij_{(t_k)}}, a^U_{ij_{(t_k)}}] \in \widehat{\alpha}_{ij}(t_k) \\ [b^L_{ij_{(t_k)}}, b^U_{ij_{(t_k)}}] \in \widehat{\beta}_{ij}(t_k) \\ [c^L_{ij_{(t_k)}}, c^U_{ij_{(t_k)}}] \in \widehat{\gamma}_{ij}(t_k)}} \left\{ \begin{matrix} \left\{ \left[a^L_{ij_{(t_k)}}, a^U_{ij_{(t_k)}} \right] \right\} \\ \left\{ \left[b^L_{ij_{(t_k)}}, b^U_{ij_{(t_k)}} \right] \right\} \\ \left\{ \left[c^L_{ij_{(t_k)}}, c^U_{ij_{(t_k)}} \right] \right\} \end{matrix} \right\} \tag{5.1}$$

for $i = 1, 2, ..., n$, $j = 1, 2, ..., m$ and $k = 1, 2, ..., r$.

Algorithm 1.

Step 1. Construct the decision matrix $D(t_k) = (d_{ij}(t_k))_{n \times m}$ and construct the idealized decision matrix $\mathfrak{D}(t_k) = (\mathfrak{d}_{ij}(t_k))_{n \times m}$.

Generally, there are two types for the criteria: benefit criteria (such criterion has a positive effect on the decision and the value of IVPHFE(s) is taken as is) and cost criteria (such criterion has a negative effect on the decision and the complement of the value of IVPHFE(s) is taken to include this criterion in the decision). We may transform the criterion values from cost type to benefit type: in such a case, the IVPHF decision matrix $D(t_k) = (d_{ij}(t_k))_{n \times m}$ can be transformed into matrix

$$\mathfrak{D}(t_k) = \left(\mathfrak{d}_{ij}(t_k)\right)_{n \times m} \text{ where for } i = 1, 2, ..., n, \ j = 1, 2, ..., m \text{ and } k = 1, 2, ..., r$$

$$\mathfrak{d}_{ij}(t_k) = \bigcup_{\substack{[\tilde{a}^L_{ij_{(t_k)}}, \tilde{a}^U_{ij_{(t_k)}}] \in \tilde{\alpha}_{ij}(t_k) \\ [\tilde{b}^L_{ij_{(t_k)}}, \tilde{b}^U_{ij_{(t_k)}}] \in \tilde{\beta}_{ij}(t_k) \\ [\tilde{c}^L_{ij_{(t_k)}}, \tilde{c}^U_{ij_{(t_k)}}] \in \tilde{\gamma}_{ij}(t_k)}} \left\{ \begin{array}{l} \{[\tilde{a}^L_{ij_{(t_k)}}, \tilde{a}^U_{ij_{(t_k)}}]\} \\ \{[\tilde{b}^L_{ij_{(t_k)}}, \tilde{b}^U_{ij_{(t_k)}}]\} \\ \{[\tilde{c}^L_{ij_{(t_k)}}, \tilde{c}^U_{ij_{(t_k)}}]\} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \displaystyle\bigcup_{\substack{[a^L_{ij_{(t_k)}}, a^U_{ij_{(t_k)}}] \in \hat{\alpha}_{ij}(t_k) \\ [b^L_{ij_{(t_k)}}, b^U_{ij_{(t_k)}}] \in \hat{\beta}_{ij}(t_k) \\ [c^L_{ij_{(t_k)}}, c^U_{ij_{(t_k)}}] \in \hat{\gamma}_{ij}(t_k)}} \left\{ \begin{array}{l} \{[a^L_{ij_{(t_k)}}, a^U_{ij_{(t_k)}}]\} \\ \{[b^L_{ij_{(t_k)}}, b^U_{ij_{(t_k)}}]\} \\ \{[c^L_{ij_{(t_k)}}, c^U_{ij_{(t_k)}}]\} \end{array} \right\}, \text{ if } \ell_j \text{ is the benefit criterion} \\[2em] \displaystyle\bigcup_{\substack{[a^L_{ij_{(t_k)}}, a^U_{ij_{(t_k)}}] \in \hat{\alpha}_{ij}(t_k) \\ [b^L_{ij_{(t_k)}}, b^U_{ij_{(t_k)}}] \in \hat{\beta}_{ij}(t_k) \\ [c^L_{ij_{(t_k)}}, c^U_{ij_{(t_k)}}] \in \hat{\gamma}_{ij}(t_k)}} \left\{ \begin{array}{l} \{[c^L_{ij_{(t_k)}}, c^U_{ij_{(t_k)}}]\} \\ \{[b^L_{ij_{(t_k)}}, b^U_{ij_{(t_k)}}]\} \\ \{[a^L_{ij_{(t_k)}}, a^U_{ij_{(t_k)}}]\} \end{array} \right\}, \text{ if } \ell_j \text{ is the cost criterion} \end{array} \right. \tag{5.2}$$

Step 2. Utilize the dynamic interval-valued picture hesitant fuzzy aggregation operators (DIVPHFWA, DIVPHFWG, GDIVPHFWA, GDIVPHFWG) to aggregate all the idealized decision matrix $\left(\mathfrak{d}_{ij}(t_k)\right)_{n \times m}$ $(k = 1, 2, ..., r)$ into the collective decision matrix $\left(\mathfrak{d}_{ij}\right)_{n \times m}$.

If we use the GDIVPHFWG operator then we take for $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$

$$\mathfrak{d}_{ij} = GDIVPHFWG_{\omega(t)}\left(\mathfrak{d}_{ij}(t_k); k = 1, 2, ..., r\right)$$

$$= \bigcup_{\substack{[\tilde{a}^L_{ij_{(t_k)}}, \tilde{a}^U_{ij_{(t_k)}}] \in \tilde{\alpha}_{ij}(t_k) \ (k=1,2,...,r) \\ [\tilde{b}^L_{ij_{(t_k)}}, \tilde{b}^U_{ij_{(t_k)}}] \in \tilde{\beta}_{ij}(t_k) \ (k=1,2,...,r) \\ [\tilde{c}^L_{ij_{(t_k)}}, \tilde{c}^U_{ij_{(t_k)}}] \in \tilde{\gamma}_{ij}(t_k) \ (k=1,2,...,r)}} \left\{ \begin{array}{l} \left\{ \left[\left(1 - \left(1 - \prod\limits_{k=1}^{r}\left(1 - \left(1 - \tilde{a}^L_{ij_{(t_k)}}\right)^\lambda\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \right), \left(1 - \left(1 - \prod\limits_{k=1}^{r}\left(1 - \left(1 - \tilde{a}^U_{ij_{(t_k)}}\right)^\lambda\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}\right) \right] \right\} \\[1em] \left\{ \left[\left(1 - \prod\limits_{k=1}^{r}\left(1 - \left(\tilde{b}^L_{ij_{(t_k)}}\right)^\lambda\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, \left(1 - \prod\limits_{k=1}^{r}(1 - (\tilde{b}^U_{ij_{(t_k)}})^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \right] \right\} \\[1em] \left\{ \left[\left(1 - \prod\limits_{k=1}^{r}\left(1 - \left(\tilde{c}^L_{ij_{(t_k)}}\right)^\lambda\right)^{\omega(t_k)}\right)^{\frac{1}{\lambda}}, \left(1 - \prod\limits_{k=1}^{r}(1 - (\tilde{c}^U_{ij_{(t_k)}})^\lambda)^{\omega(t_k)}\right)^{\frac{1}{\lambda}} \right] \right\} \end{array} \right\}. \tag{5.3}$$

Step 3. Utilize the interval-valued picture hesitant fuzzy aggregation operators (IVPHFWA, IVPHFWG) to find the collective overall preference values $\mathfrak{d}_i$ $(i = 1, 2, ..., n)$ of the alternatives q_i $(i = 1, 2, ..., n)$.

If we use the IVPHFWG operator then we take for $i = 1, 2, ..., n$

$$\mathfrak{d}_i = IVPHFWG_\omega\left(\mathfrak{d}_{ij}; j = 1, 2, ..., m\right)$$

$$= \bigcup_{\substack{[\tilde{a}^L_{ij}, \tilde{a}^U_{ij}] \in \tilde{\alpha}_{ij} \ (j=1,2,...,m) \\ [\tilde{b}^L_{ij}, \tilde{b}^U_{ij}] \in \tilde{\beta}_{ij} \ (j=1,2,...,m) \\ [\tilde{c}^L_{ij}, \tilde{c}^U_{ij}] \in \tilde{\gamma}_{ij} \ (j=1,2,...,m)}} \left\{ \begin{array}{l} \left\{ \left[\prod\limits_{j=1}^{m}\left(\tilde{a}^L_{ij}\right)^{\omega_j}, \prod\limits_{j=1}^{m}\left(\tilde{a}^U_{ij}\right)^{\omega_j} \right] \right\} \\[1em] \left\{ \left[1 - \prod\limits_{j=1}^{m}\left(1 - \tilde{b}^L_{ij}\right)^{\omega_j}, 1 - \prod\limits_{j=1}^{m}\left(1 - \tilde{b}^U_{ij}\right)^{\omega_j} \right] \right\} \\[1em] \left\{ \left[1 - \prod\limits_{j=1}^{m}\left(1 - \tilde{c}^L_{ij}\right)^{\omega_j}, 1 - \prod\limits_{j=1}^{m}\left(1 - \tilde{c}^U_{ij}\right)^{\omega_j} \right] \right\} \end{array} \right\}. \tag{5.4}$$

Step 4. Calculate the score values $\mathcal{S}(\mathfrak{d}_i)$ of $\mathfrak{d}_i$ ($i = 1, 2, ..., n$).

When two score values $\mathcal{S}(\mathfrak{d}_i)$ and $\mathcal{S}(\mathfrak{d}_{i'})$ are equal, calculate the accuracy values $\mathcal{A}(\mathfrak{d}_i)$ and $\mathcal{A}(\mathfrak{d}_{i'})$. If $\mathcal{A}(\mathfrak{d}_i)$ and $\mathcal{A}(\mathfrak{d}_{i'})$ are qual, calculate the certainty values $\mathcal{C}(\mathfrak{d}_i)$ and $\mathcal{C}(\mathfrak{d}_{i'})$.

Step 5. Rank all the alternatives $\{q_1, q_2, ..., q_n\}$ and select the optimal one(s) by the score values (the accuracy values in the case of equality of score values, the certainty values in the case of equality of accuracy values).

Algorithm 2.

Step 1. Same as Step 1 of Algorithm 1.

Step 2. Utilize the dynamic interval-valued picture hesitant fuzzy Einstein aggregation operators (DIVPHFEWA, DIVPHFEWG, GDIVPHFEWA, GDIVPHFEWG) to aggregate all the idealized decision matrix $\left(\mathfrak{d}_{ij}(t_k)\right)_{n\times m}$ ($k = 1, 2, ..., r$) into the collective decision matrix $\left(\mathfrak{d}_{ij}\right)_{n\times m}$.

If we use the GDIVPHFEWA operator then we take for $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$

$$\mathfrak{d}_{ij} = GDIVPHFEWA_{\omega(t)}\left(\mathfrak{d}_{ij}(t_k); k = 1, 2, ..., r\right)$$

$$
= \bigcup_{\substack{[\tilde{a}^L_{ij_{(t_k)}}, \tilde{a}^U_{ij_{(t_k)}}]\in \widetilde{\alpha}_{ij}(t_k)\ (k=1,2,...,r) \\ [\tilde{b}^L_{ij_{(t_k)}}, \tilde{b}^U_{ij_{(t_k)}}]\in \widetilde{\beta}_{ij}(t_k)\ (k=1,2,...,r) \\ [\tilde{c}^L_{ij_{(t_k)}}, \tilde{c}^U_{ij_{(t_k)}}]\in \widetilde{\gamma}_{ij}(t_k)\ (k=1,2,...,r)}}
\left\{ \left\{ \left[\frac{2\left(\prod_{k=1}^{r}(\eta_{\tilde{a}^L_{ij_{(t_k)}}})^{\omega(t_k)} - \prod_{k=1}^{r}(\vartheta_{\tilde{a}^L_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}(\eta_{\tilde{a}^L_{ij_{(t_k)}}})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\vartheta_{\tilde{a}^L_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}(\eta_{\tilde{a}^L_{ij_{(t_k)}}})^{\omega(t_k)} - \prod_{k=1}^{r}(\vartheta_{\tilde{a}^L_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}} , \right. \right. \right.
$$

$$
\left. \frac{2\left(\prod_{k=1}^{r}(\eta_{\tilde{a}^U_{ij_{(t_k)}}})^{\omega(t_k)} - \prod_{k=1}^{r}(\vartheta_{\tilde{a}^U_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}(\eta_{\tilde{a}^U_{ij_{(t_k)}}})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\vartheta_{\tilde{a}^U_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}(\eta_{\tilde{a}^U_{ij_{(t_k)}}})^{\omega(t_k)} - \prod_{k=1}^{r}(\vartheta_{\tilde{a}^U_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}} \right],
$$

$$
\left[\frac{\left(\prod_{k=1}^{r}(\zeta_{\tilde{b}^L_{ij_{(t_k)}}})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\theta_{\tilde{b}^L_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{r}(\zeta_{\tilde{b}^L_{ij_{(t_k)}}})^{\omega(t_k)} - \prod_{k=1}^{r}(\theta_{\tilde{b}^L_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}(\zeta_{\tilde{b}^L_{ij_{(t_k)}}})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\theta_{\tilde{b}^L_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}(\zeta_{\tilde{b}^L_{ij_{(t_k)}}})^{\omega(t_k)} - \prod_{k=1}^{r}(\theta_{\tilde{b}^L_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}} ,
$$

$$
\frac{\left(\prod_{k=1}^{r}(\zeta_{\tilde{b}^U_{ij_{(t_k)}}})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\theta_{\tilde{b}^U_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{r}(\zeta_{\tilde{b}^U_{ij_{(t_k)}}})^{\omega(t_k)} - \prod_{k=1}^{r}(\theta_{\tilde{b}^U_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}(\zeta_{\tilde{b}^U_{ij_{(t_k)}}})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\theta_{\tilde{b}^U_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}(\zeta_{\tilde{b}^U_{ij_{(t_k)}}})^{\omega(t_k)} - \prod_{k=1}^{r}(\theta_{\tilde{b}^U_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}} \right],
$$

$$
\left[\frac{\left(\prod_{k=1}^{r}(\xi_{\tilde{c}^L_{ij_{(t_k)}}})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\psi_{\tilde{c}^L_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{r}(\xi_{\tilde{c}^L_{ij_{(t_k)}}})^{\omega(t_k)} - \prod_{k=1}^{r}(\psi_{\tilde{c}^L_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}(\xi_{\tilde{c}^L_{ij_{(t_k)}}})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\psi_{\tilde{c}^L_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}(\xi_{\tilde{c}^L_{ij_{(t_k)}}})^{\omega(t_k)} - \prod_{k=1}^{r}(\psi_{\tilde{c}^L_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}} ,
$$

$$
\left. \left. \frac{\left(\prod_{k=1}^{r}(\xi_{\tilde{c}^U_{ij_{(t_k)}}})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\psi_{\tilde{c}^U_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} - \left(\prod_{k=1}^{r}(\xi_{\tilde{c}^U_{ij_{(t_k)}}})^{\omega(t_k)} - \prod_{k=1}^{r}(\psi_{\tilde{c}^U_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}}{\left(\prod_{k=1}^{r}(\xi_{\tilde{c}^U_{ij_{(t_k)}}})^{\omega(t_k)} + 3\prod_{k=1}^{r}(\psi_{\tilde{c}^U_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}} + \left(\prod_{k=1}^{r}(\xi_{\tilde{c}^U_{ij_{(t_k)}}})^{\omega(t_k)} - \prod_{k=1}^{r}(\psi_{\tilde{c}^U_{ij_{(t_k)}}})^{\omega(t_k)}\right)^{\frac{1}{\lambda}}} \right] \right\} \right\}
$$

$$(5.5)$$

where

$$
\begin{cases}
\eta_{\tilde{a}^L_{ij_{(t_k)}}} = \left(2 - \tilde{a}^L_{ij_{(t_k)}}\right)^{\lambda} + 3\left(\tilde{a}^L_{ij_{(t_k)}}\right)^{\lambda}, & \vartheta_{\tilde{a}^L_{ij_{(t_k)}}} = \left(2 - \tilde{a}^L_{ij_{(t_k)}}\right)^{\lambda} - \left(\tilde{a}^L_{ij_{(t_k)}}\right)^{\lambda}, \\[4pt]
\eta_{\tilde{a}^U_{ij_{(t_k)}}} = \left(2 - \tilde{a}^U_{ij_{(t_k)}}\right)^{\lambda} + 3\left(\tilde{a}^U_{ij_{(t_k)}}\right)^{\lambda}, & \vartheta_{\tilde{a}^U_{ij_{(t_k)}}} = \left(2 - \tilde{a}^U_{ij_{(t_k)}}\right)^{\lambda} - \left(\tilde{a}^U_{ij_{(t_k)}}\right)^{\lambda}, \\[4pt]
\zeta_{\tilde{b}^L_{ij_{(t_k)}}} = \left(1 + \tilde{b}^L_{ij_{(t_k)}}\right)^{\lambda} + 3\left(1 - \tilde{b}^L_{ij_{(t_k)}}\right)^{\lambda}, & \theta_{\tilde{b}^L_{ij_{(t_k)}}} = \left(1 + \tilde{b}^L_{ij_{(t_k)}}\right)^{\lambda} - \left(1 - \tilde{b}^L_{ij_{(t_k)}}\right)^{\lambda}, \\[4pt]
\zeta_{\tilde{b}^U_{ij_{(t_k)}}} = \left(1 + \tilde{b}^U_{ij_{(t_k)}}\right)^{\lambda} + 3\left(1 - \tilde{b}^U_{ij_{(t_k)}}\right)^{\lambda}, & \theta_{\tilde{b}^U_{ij_{(t_k)}}} = \left(1 + \tilde{b}^U_{ij_{(t_k)}}\right)^{\lambda} - \left(1 - \tilde{b}^U_{ij_{(t_k)}}\right)^{\lambda}, \\[4pt]
\xi_{\tilde{c}^L_{ij_{(t_k)}}} = \left(1 + \tilde{c}^L_{ij_{(t_k)}}\right)^{\lambda} + 3\left(1 - \tilde{c}^L_{ij_{(t_k)}}\right)^{\lambda}, & \psi_{\tilde{c}^L_{ij_{(t_k)}}} = \left(1 + \tilde{c}^L_{ij_{(t_k)}}\right)^{\lambda} - \left(1 - \tilde{c}^L_{ij_{(t_k)}}\right)^{\lambda}, \\[4pt]
\xi_{\tilde{c}^U_{ij_{(t_k)}}} = \left(1 + \tilde{c}^U_{ij_{(t_k)}}\right)^{\lambda} + 3\left(1 - \tilde{c}^U_{ij_{(t_k)}}\right)^{\lambda}, & \psi_{\tilde{c}^U_{ij_{(t_k)}}} = \left(1 + \tilde{c}^U_{ij_{(t_k)}}\right)^{\lambda} - \left(1 - \tilde{c}^U_{ij_{(t_k)}}\right)^{\lambda}.
\end{cases}
$$

$$(5.6)$$

Step 3. Utilize the interval-valued picture hesitant fuzzy aggregation operators (IVPHFWA, IVPHFWG) to find the collective overall preference values $\eth_i$ ($i = 1, 2, ..., n$) of the alternatives q_i ($i = 1, 2, ..., n$).

If we use the IVPHFWA operator then we take for $i = 1, 2, ..., n$

$$\eth_i = IVPHFWA_\omega \left(\eth_{ij}; j = 1, 2, ..., m \right)$$

$$= \bigcup_{\substack{[\tilde{a}_{ij}^L, \tilde{a}_{ij}^U] \in \tilde{\alpha}_{ij} \ (j=1,2,...,m) \\ [\tilde{b}_{ij}^L, \tilde{b}_{ij}^U] \in \tilde{\beta}_{ij} \ (j=1,2,...,m) \\ [\tilde{c}_{ij}^L, \tilde{c}_{ij}^U] \in \tilde{\gamma}_{ij} \ (j=1,2,...,m)}} \left\{ \begin{array}{l} \left\{ \left[1 - \prod_{j=1}^{m} (1 - \tilde{a}_{ij}^L)^{\omega_j}, 1 - \prod_{j=1}^{m} (1 - \tilde{a}_{ij}^U)^{\omega_j} \right] \right\} \\ \left\{ \left[\prod_{j=1}^{m} (\tilde{b}_{ij}^L)^{\omega_j}, \prod_{j=1}^{m} (\tilde{b}_{ij}^U)^{\omega_j} \right] \right\} \\ \left\{ \left[\prod_{j=1}^{m} (\tilde{c}_{ij}^L)^{\omega_j}, \prod_{j=1}^{m} (\tilde{c}_{ij}^U)^{\omega_j} \right] \right\} \end{array} \right\}. \quad (5.7)$$

Step 4. Same as Step 4 of Algorithm 1.

Step 5. Same as Step 5 of Algorithm 1.

To clarify the organization of the proposed algorithms, we present the the combined frame diagram of Algorithm 1 and Algorithm 2 in Fig. 2.

Example 5.1 (For Algorithm 1) Suppose that a high-level committee has been formed to investigate the impacts of COVID-19 in daily life for the four major cities q_1, q_2, q_3, and q_4 (extendable) in a country. This committee consists of COVID-19 board members appointed by the ministry of health, economy and society (government departments). The committee is directed to explore the cities based on the three most important parameters l_1-Healthcare, l_2-Economic, and l_3-Social throughout the two significant lockdown periods t_1 and t_2. The committee is analyzing the Challenges in the diagnosis, Quarantine, and remedy of suspected or confirmed cases, High burden of the functioning of the present medical system, Patients with different disorder and fitness troubles are getting neglected, Overload on doctors and different fitness care professionals, Overloading of medical shops, and Disruption of the medical supply chain under the parameter l_1-Healthcare. Slowing the production of crucial goods, Disrupt the supply chain of products, Loss in countrywide and global business, Poor cash flow in the market, and Significant slowing down in the sales boom under the parameter l_2-Economic. Service quarter isn't always being capable of offering their right service, Cancellation or postponement of large-scale sports activities and tournaments, Avoiding the countrywide and global traveling and cancellation of services, Disruption of the celebration of cultural, spiritual, and festive events, Undue strain most of the population, Social distancing with our friends and own circle of relatives members, Closure of the hotels, eating places and religious places, Closure of places for leisure which include film and play theaters, sports activities, clubs, gymnasiums, and swimming pools, Postponement of examinations under the parameter l_3-Social. Assume that $\omega = (0.2, 0.3, 0.5)$ be the weightage of criteria l_1, l_2 and l_3 and $\omega(t) = (0.6, 0.4)$ be the weightage of the time periods t_1 and t_2. Suppose that experts create the idealized decision matrix table (see Table 1) which has the dynamic interval-valued picture hesitant fuzzy variables/numbers. We can find the mostly affected city in daily life by COVID-19. It is being executed through Step 1 to Step 5 of Algorithm 1 as follows.

The values provided in Table 1 consistent with the committee evaluations for the effects of COVID-19 can be interpreted as follows. From the topmost access {{[0.15, 0.90]}, {[0.01, 0.02], [0.02, 0.03]}, {[0.04, 0.05]}} of Table 1, we apprehend that $4-5\%$ of Healthcare sector is affected, $1-2\%$ or $2-3\%$ of Healthcare sector is neither affected, nor benefited, and
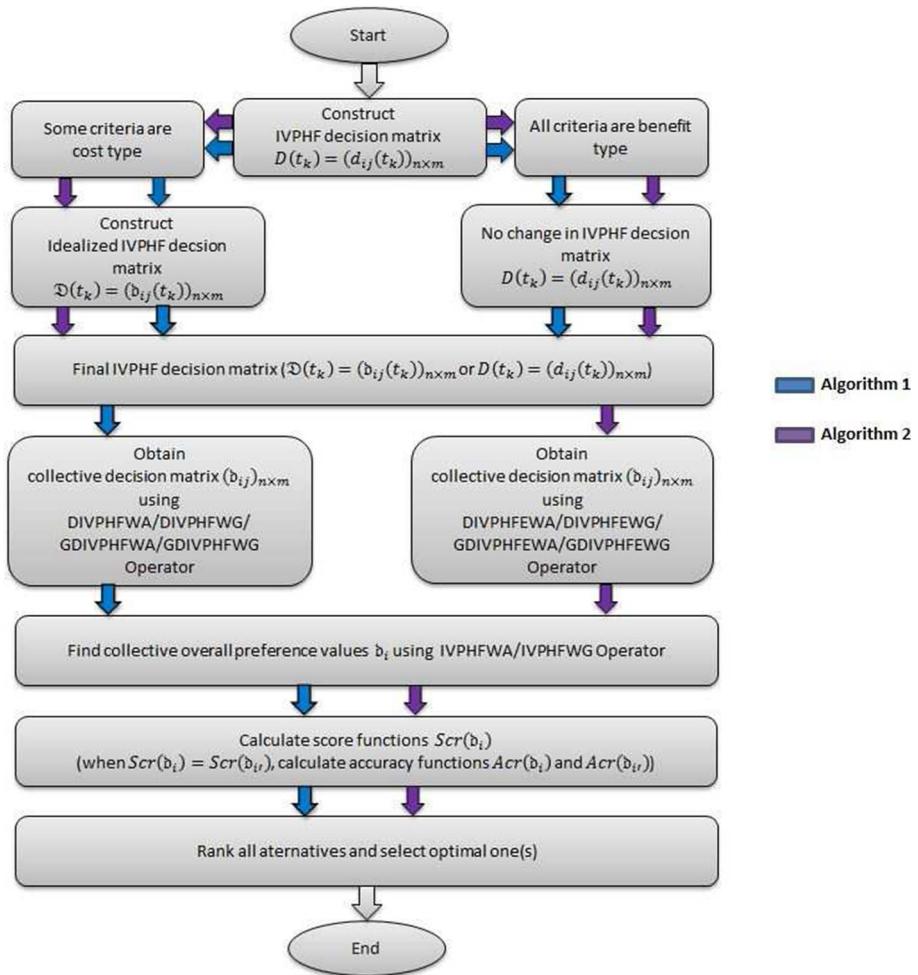
Fig. 2 Frame diagram of Algorithm 1 and Algorithm 2

$15-90\%$ of Healthcare sector is not affected during the period t_1 for the city q_1 due to COVID-19. From the second one topmost access $\{\{[0.08, 0.80]\}, \{[0.01, 0.02]\}, \{[0.14, 0.16]\}\}$ of Table 1, we recognize that $14-16\%$ of Economic sector is affected, $1-2\%$ of Economic sector is neither affected, nor benefited, and $8-80\%$ of Economic sector is not affected during the period t_1 for the city q_1 due to COVID-19. From the third topmost access $\{\{[0.07, 0.70]\}, \{[0.10, 0.11]\}, \{[0.01, 0.02]\}\}$ of Table 1, we understand that $1-2\%$ of Social sector is affected, $10-11\%$ of Social sector is neither affected nor benefited, and $7-70\%$ of Social sector is not affected during the period t_1 for the city q_1 due to COVID-19. Similarly, we can interpret all of the values of Table 1 as well. We can find the mostly affected city due to COVID-19 through executing the Step 1 to Step 5 of Algorithm 1 as follows.

Step 1. The idealized decision matrix $\mathfrak{D}(t_k) = \big(\eth_{ij}(t_k)\big)_{4\times 3}$ $(i = 1, 2, 3, 4, j = 1, 2, 3, k = 1, 2)$ is constructed from Table 1.

Table 1 Table for idealized decision matrix

Time-moment	Para-meter	Alternative q_1
t_1	l_1	{{[0.15, 0.90]}, {[0.01, 0.02], [0.02, 0.03]}, {[0.04, 0.05]}}
	l_2	{{[0.08, 0.80]}, {[0.01, 0.02]}, {[0.14, 0.16]}}
	l_3	{{[0.07, 0.70]}, {[0.10, 0.11]}, {[0.01, 0.02]}}
t_2	l_1	{{[0.04, 0.85]}, {[0.01, 0.03]}, {[0.01, 0.03], [0.07, 0.08]}}
	l_2	{{[0.01, 0.70], [0.08, 0.81]}, {[0.08, 0.09]}, {[0.01, 0.03]}}
	l_3	{{[0.07, 0.65]}, {[0.09, 0.10]}, {[0.08, 0.10]}}

q_2

{{[0.04, 0.30]}, {[0.22, 0.33], [0.28, 0.40]}, {[0.28, 0.30]}}
{{[0.13, 0.50], [0.41, 0.48]}, {[0.11, 0.20]}, {[0.18, 0.30]}}
{{[0.18, 0.30]}, {[0.12, 0.18]}, {[0.17, 0.50]}}
{{[0.17, 0.50]}, {[0.11, 0.30]}, {[0.08, 0.20]}}
{{[0.27, 0.38]}, {[0.23, 0.40]}, {[0.14, 0.20], [0.11, 0.12]}}
{{[0.05, 0.60]}, {[0.04, 0.20]}, {[0.01, 0.20]}}

q_3

{{[0.31, 0.50]}, {[0.11, 0.14], [0.04, 0.20]}, {[0.21, 0.28]}}
{{[0.28, 0.40]}, {[0.23, 0.40]}, {[0.01, 0.20]}}
{{[0.01, 0.20], [0.17, 0.18]}, {[0.20, 0.30]}, {[0.21, 0.33]}}
{{[0.03, 0.20]}, {[0.33, 0.48]}, {[0.01, 0.20]}}
{{[0.31, 0.50]}, {[0.13, 0.18]}, {[0.08, 0.20]}}
{{[0.11, 0.20]}, {[0.22, 0.40]}, {[0.14, 0.40], [0.11, 0.33]}}

q_4

{{[0.11, 0.50]}, {[0.11, 0.30]}, {[0.05, 0.20]}}
{{[0.17, 0.30]}, {[0.11, 0.30], [0.11, 0.14]}, {[0.17, 0.40]}}
{{[0.61, 0.80]}, {[0.08, 0.10]}, {[0.01, 0.10], [0.02, 0.09]}}
{{[0.14, 0.20]}, {[0.17, 0.20]}, {[0.41, 0.60]}}
{{[0.26, 0.50]}, {[0.13, 0.26]}, {[0.01, 0.10]}}
{{[0.22, 0.40], [0.18, 0.28]}, {[0.13, 0.30]}, {[0.04, 0.30]}}

Step 2. The collective decision matrix $(\mathfrak{d}_{ij})_{4\times3}$ ($i = 1, 2, 3, 4$, $j = 1, 2, 3$) is constructed from Table 2 by utilizing the dynamic interval-valued picture hesitant fuzzy weighted average (DIVPHFWA) operator (Algorithm 1).

Step 3. The collective overall preference values $\mathfrak{d}_i$ ($i = 1, 2, 3, 4$) of the alternatives q_i ($i = 1, 2, 3, 4$) are obtained from Table 3 by utilizing the interval-valued picture hesitant fuzzy weighted average (IVPHFWA) operator.

Step 4. The score functions $\mathcal{S}(\mathfrak{d}_i)$ of $\mathfrak{d}_i$ ($i = 1, 2, 3, 4$) are obtained as mentioned in Table 4. Therefore, we get the ranking order of given alternatives is $q_1 \succ q_4 \succ q_2 \succ q_3$ according to DIVPHFWA operator.

If we apply DIVPHFWG operator, we get the score values and ranking order of alternatives as mentioned in Table 4. The score values and ranking order of alternatives using the GDIVPHFWA and GDIVPHFWG operators for different values of λ are displayed in Table 5. By considering the dynamic aggregation operators: DIVPHFWA and DIVPHFWG and the generalized dynamic aggregation operators: GDIVPHFWA
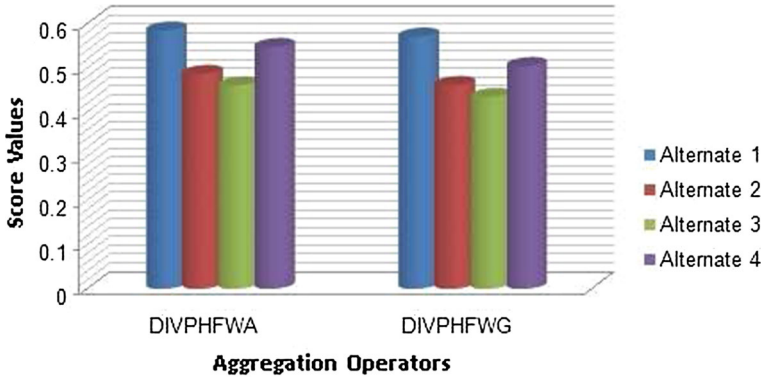
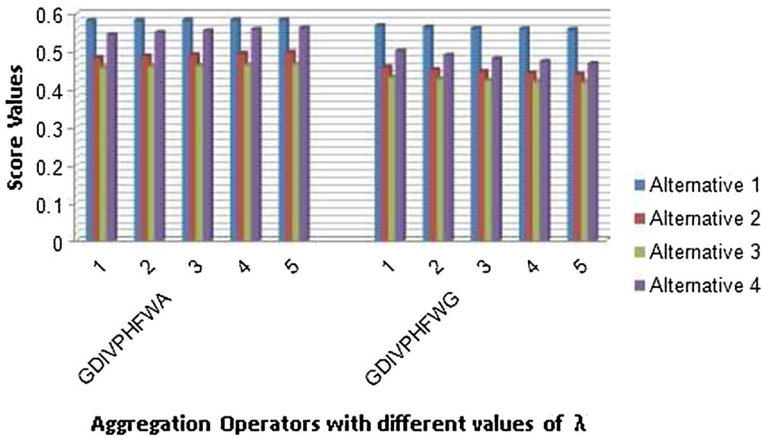**Fig. 3** Comparison chart for DIVPHFWA–DIVPHFWG operators

Fig. 4 Comparison chart for GDIVPHFWA–GDIVPHFWG operators with different values of λ

and GDIVPHFWG, we point out that the score values for the alternatives are different, but the ranking order of the alternatives is the same. It is clearly visible in Figs. 3 and 4.

Example 5.2 (For Algorithm 2) To illustrate the steps of Algorithm 2, we deal with the investment opportunities selection problem. A fund manager Mr. M in a wealth management firm is assessing three potential investment opportunities q_1, q_2, q_3. The firm mandates that the fund manager has to evaluate the following three parameters l_1 and l_2, where l_1 stands for the 'risk' and l_2 stands for the 'growth'. For each parameter l_i $(i = 1, 2)$, Mr. M invites two expert groups to evaluate the q_j $(j = 1, 2, 3)$ with respect to l_i $(i = 1, 2)$ in three different periods of time t_1, t_2, t_3. Assume that $\omega = (0.3, 0.7)$ be the weightage of criteria l_1 and l_2 and $\omega(t) = (0.1, 0.3, 0.6)$ be the weightage of the time periods t_1, t_2, and t_3. Suppose that experts create the idealized decision matrix table (see Table 6) which has the dynamic interval-valued picture hesitant fuzzy variables/numbers. The most suitable parameter can be selected for investment through the Step 1 to Step 5 of Algorithm 2 as follows.

Step 1. The idealized decision matrix $\mathfrak{D}(t_k) = \left(\mathfrak{d}_{ij}(t_k)\right)_{3\times 2}$ $(i = 1, 2, 3, j = 1, 2, k = 1, 2, 3)$ is constructed from Table 6.

Table 2 Table for collective decision matrix

Para-meter	Alternative q1
l1	{{[0.1075994, 0.8823921]}, {[0.01, 0.0235216], [0.0151572, 0.03]}, {[0.022974, 0.04407597], [0.0500352, 0.0603418]}}
l2	{{[0.0526144, 0.7647842], [0.08, 0.8040617]}, {[0.022974, 0.0365019]}, {[0.0487166, 0.0819069]}}
l3	{{[0.07, 0.6809197]}, {[0.0958732, 0.1058853]}, {[0.022974, 0.03807311], }}
	q2
l1	{{[0.09428, 0.3881474]}, {[0.1667288, 0.3176559], [0.1926868, 0.3565205]}, {[0.169641, 0.2550849]}}
l2	{{[0.1889629, 0.4550726], [0.3575485, 0.442097]}, {[0.1477497, 0.2639016], {[0.1627851, 0.2550849], [0.1478157, 0.2079435]}}
l3	{{[0.1302834, 0.4403934]}, {[0.0773273, 0.1877481]}, {[0.0547355, 0.3465724]}}
	q3
l1	{{[0.2092883, 0.3965824]}, {[0.170703, 0.2291782], [0.093034, 0.2838667]}, {[0.0621343, 0.244741]}}
l2	{{[0.2921534, 0.4421997]}, {[0.1830687, 0.290633]}, {[0.022974, 0.2]}}
l3	{{[0.051282, 0.2], [0.1465013, 0.1880593]}, {[0.207772, 0.3365865]}, {[0.1785594, 0.3563956], [0.1621394, 0.33]}}
	q4
l1	{{[0.1221236, 0.3965824]}, {[0.1309228, 0.2550849]}, {[0.11601, 0.3103691]}}
l2	{{[0.207244, 0.3881474]}, {[0.1176015, 0.2833101], [0.1176015, 0.1793355]}, {[0.0547355, 0.2297397]}}
l3	{{[0.4853919, 0.6896309], [0.474994, 0.6661503]}, {[0.0971474, 0.1551846], [0.017411, 0.1551846], [0.0263902, 0.145678]}}

Table 3 Table for collective overall preference values

q_1

{{[[0.0725052, 0.7615061], [0.0806311, 0.7742264]], {[[0.0397392, 0.0569384], [0.043186, 0.0597773]], {[[0.0287851, 0.04856811], [0.0336338, 0.0525326]]}}

q_2

{{[[0.1413822, 0.4348387], [0.1993554, 0.4308347]], {[[0.1095031, 0.2309999], [0.1127184, 0.2363944]], {[[0.0951755, 0.2973304], [0.0924607, 0.2796517]]}}

q_3

{{[[0.1621738, 0.3214026], [0.20533, 0.316357]], {[[0.1923207, 0.29825421], [0.1703358, 0.3112969]], {[[0.0781499, 0.2779819], [0.07447, 0.2674899]]}}

q_4

{{[[0.3481212, 0.5654328], [0.3415684, 0.5492942]], {[[0.109205, 0.20532411], [0.109205, 0.1790033]], {[[0.035875, 0.2005261], [0.0441673, 0.1942869]]}}

Table 4 Table of score values-ranking order for DIVPHFWA–DIVPHFWG operators

Operators	Score values				Ranking order
DIVPHFWA	0.5828568	0.4845111	0.4584353	0.5454265	$q_1 \succ q_4 \succ q_2 \succ q_3$
DIVPHFWG	0.569065	0.4593005	0.432204	0.50235	$q_1 \succ q_4 \succ q_2 \succ q_3$

Table 5 Ranking order of alternatives for GDIVPHFWA–GDIVPHFWG operators with different values of λ

Operators	λ	Score values				Ranking order
	1	0.5828568	0.4845111	0.4584353	0.5454265	$q_1 \succ q_4 \succ q_2 \succ q_3$
	2	0.5834922	0.4885244	0.4613326	0.5508532	$q_1 \succ q_4 \succ q_2 \succ q_3$
GDIVPHFWA	3	0.5839548	0.4923313	0.4635947	0.5556214	$q_1 \succ q_4 \succ q_2 \succ q_3$
	4	0.5843024	0.4957437	0.4654296	0.5596157	$q_1 \succ q_4 \succ q_2 \succ q_3$
	5	0.5845858	0.4987449	0.4669893	0.5629243	$q_1 \succ q_4 \succ q_2 \succ q_3$
	1	0.569065	0.4593005	0.432204	0.50235	$q_1 \succ q_4 \succ q_2 \succ q_3$
	2	0.5650235	0.4530838	0.4278411	0.4910611	$q_1 \succ q_4 \succ q_2 \succ q_3$
GDIVPHFWG	3	0.5623685	0.4479962	0.424282	0.4816923	$q_1 \succ q_4 \succ q_2 \succ q_3$
	4	0.5606428	0.4439221	0.4214459	0.4744388	$q_1 \succ q_4 \succ q_2 \succ q_3$
	5	0.5594333	0.4406658	0.4191625	0.4688665	$q_1 \succ q_4 \succ q_2 \succ q_3$

Table 6 Table for idealized decision matrix

Time–moment	Parameter	Alternative q_1
t_1	l_1	{{[0.05, 0.10]}, {[0.10, 0.30]}, {[0.12, 0.20], [0.10, 0.30]}}
	l_2	{{[0.14, 0.30]}, {[0.21, 0.40]}, {[0.15, 0.30]}}
t_2	l_1	{{[0.14, 0.40]}, {[0.03, 0.10]}, {[0.02, 0.03], [0.01, 0.10]}}
	l_2	{{[0.21, 0.30], [0.18, 0.21]}, {[0.32, 0.40]}, {[0.01, 0.10]}}
t_3	l_1	{{[0.22, 0.40]}, {[0.03, 0.20]}, {[0.01, 0.04]}}
	l_2	{{[0.21, 0.50]}, {[0.05, 0.10]]}, {[0.03, 0.20]}}

q_2

{{[0.14, 0.30]}, {[0.02, 0.13], [0.08, 0.20]}, {[0.18, 0.20]}}

{{[0.02, 0.10]}, {[0.01, 0.10]}, {[0.19, 0.20]}}

{{[0.16, 0.50]}, {[0.01, 0.20]}, {[0.05, 0.10]}}

{{[0.17, 0.18], [0.16, 0.30]}, {[0.04, 0.20]}, {[0.15, 0.30]}}

{{[0.13, 0.30]}, {[0.13, 0.28]}, {[0.01, 0.10]}}

{{[0.21, 0.40]}, {[0.13, 0.28]}, {[0.08, 0.30]}}

q_3

{{[0.01, 0.20]}, {[0.04, 0.30]}, {[0.11, 0.18], [0.04, 0.20]}}

{{[0.18, 0.20]}, {[0.03, 0.10]}, {[0.02, 0.21]}}

{{[0.03, 0.10]}, {[0.13, 0.18], [0.21, 0.40]}, {[0.01, 0.10]}}

{{[0.21, 0.40]}, {[0.13, 0.18], [0.12, 0.27]}, {[0.08, 0.10]}}

{{[0.03, 0.10]}, {[0.13, 0.28]}, {[0.01, 0.30]}}
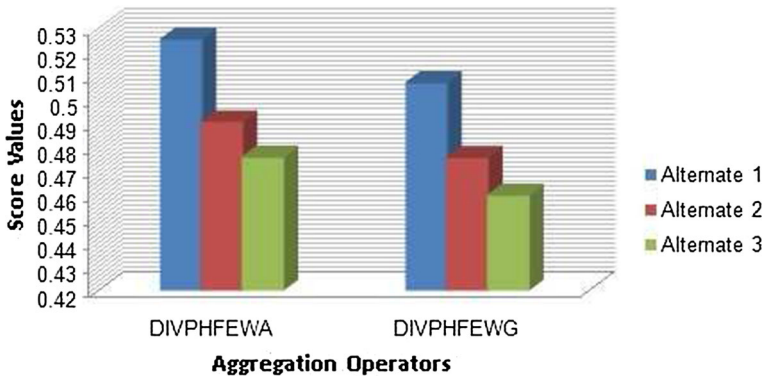
{{[0.01, 0.20]}, {[0.03, 0.28]}, {[0.08, 0.10]}}

Fig. 5 Comparison chart for DIVPHFEWA–DIVPHFEWG operators

Step 2. The collective decision matrix $(\mathfrak{d}_{ij})_{3\times 2}$ ($i = 1, 2, 3$, $j = 1, 2$) is constructed from Table 7 by utilizing the interval-valued picture hesitant fuzzy Einstein weighted average (IVPHFEWA) operator (Algorithm 2).

Step 3. The collective overall preference values $\mathfrak{d}_i$ ($i = 1, 2, 3$) of the alternatives q_i ($i = 1, 2, 3$) are obtained from Table 8 by utilizing the interval-valued picture hesitant fuzzy weighted average (IVPHFWA) operator.

Step 4. The score functions $\mathcal{S}(\mathfrak{d}_i)$ of $\mathfrak{d}_i$ ($i = 1, 2, 3$) are obtained as mentioned in Table 9. Therefore, we get the ranking order of given alternatives is $q_1 \succ q_2 \succ q_3$ according to DIVPHFEWA operator.

If we apply DIVPHFEWG operator, we get the score values and ranking order of alternatives as mentioned in Table 9. The score values and ranking order of alternatives using the GDIVPHFEWA and GDIVPHFEWG operators for different values of λ are displayed in Table 10. By considering the dynamic aggregation operators based on Einstein operations: DIVPHFEWA and DIVPHFEWG and the generalized dynamic aggregation operators based on Einstein operations: DIVPHFEWA and DIVPHFEWG, we point out that the score values for the alternatives are different, but the ranking order of the alternatives is the same. It is visible through Figs. 5 and 6.

Limitations of the proposed approaches:

To reveal the limitations of the proposed approaches, we analyze Algorithm 1 and Algorithm 2 critically and give their drawbacks as follows.

In the above examples, possible dependencies between criteria (parameters) are neglected. It is practically not always possible to assume in the MPDM that each parameter is independent of other parameters. Any parameter in the MPDM could be related to, or dependent on other parameters. In the proposed MPDM approaches, evaluating the dependencies among parameters should contribute to the objectivity of decisions. The considering dependency in the interval-valued picture hesitant fuzzy MPDM may improve the quality of decision-making process.

Table 7 Table for collective decision matrix

Parameter	Alternative q_1
l_1	{{[0.1795092, 0.3724955]}, {[0.0338939, 0.1701784]}, {[0.0158516, 0.0433284], [[0.0128754, 0.0622487], [0.0155511, 0.0453335], [0.0126309, 0.0651002]}}
l_2	{{[0.2030837, 0.4246829], [0.1940961, 0.4007038]}, {[0.1033335, 0.1787678]}, {[0.0254853, 0.1701784]}}
	q_2
	{{[0.1400264, 0.3639709]}, {[0.0508039, 0.2352829], [0.0583127, 0.245194]}, {[0.0218349, 0.107342]}}
	{{[0.1794391, 0.3084375], [0.1764528, 0.3426383]}, {[0.0713779, 0.2295668]}, {[0.105707, 0.2883931]}}
	q_3
	{{[0.0280008, 0.1101266]}, {[0.115873, 0.2476439], [0.1342473, 0.3145825]}, {[0.0127578, 0.2076645], [0.0114958, 0.2098212]}}
	{{[0.0879223, 0.262701]}, {[0.0469043, 0.2225648], [0.0457453, 0.2511081]}, {[0.069799, 0.1078955]}}

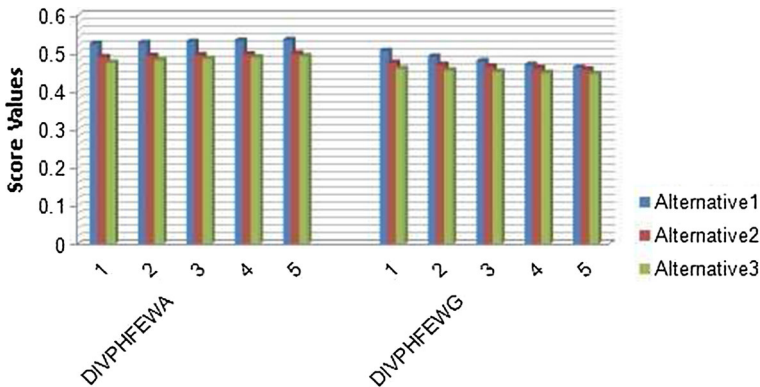**Table 8** Table for collective overall preference values

q_1

{{[[0.1960833, 0.4094996], [0.1897475, 0.3923771]], {[[0.0739614, 0.1761464], {[[0.0221017, 0.1128924], [0.0207649, 0.1258555], [0.0219751, 0.114435], [0.0206459, 0.127558]]}}

q_2

{{[[0.1678088, 0.3255883], [0.1656899, 0.3491122]], {[[0.0644561, 0.2312669], [0.0671775, 0.2341474]], [0.0658594, 0.2143978]]}}

q_3

{{[[0.0703444, 0.2199023]], {[[0.0615238, 0.2298094], [0.0604557, 0.2500638], [0.0643014, 0.2469107], [0.063185, 0.2686723]], {[0.0419206, 0.1313103], [0.0406309, 0.1317217]]}}

Table 9 Table of score values-ranking order for DIVPHFEWA–DIVPHFEWG operators

Operators	Score values			Ranking order
DIVPHFEWA	0.5252736	0.4906648	0.4757781	$q_1 \succ q_2 \succ q_3$
DIVPHFEWG	0.5070679	0.4755731	0.4598301	$q_1 \succ q_2 \succ q_3$

Table 10 Ranking order alternatives for GDIVPHFEWA–GDIVPHFEWG operators with different values of λ

Operators	λ	Score functions			Ranking Order
	1	0.5252736	0.4906648	0.4757781	$q_1 \succ q_2 \succ q_3$
	2	0.5279883	0.4937191	0.4819029	$q_1 \succ q_2 \succ q_3$
GDIVPHFEWA	3	0.5307157	0.4960927	0.4861892	$q_1 \succ q_2 \succ q_3$
	4	0.5331643	0.4980722	0.4895613	$q_1 \succ q_2 \succ q_3$
	5	0.5353295	0.4998603	0.4924096	$q_1 \succ q_2 \succ q_3$
	1	0.5070679	0.4755731	0.4598301	$q_1 \succ q_2 \succ q_3$
	2	0.4918949	0.4703409	0.4553754	$q_1 \succ q_2 \succ q_3$
GDIVPHFEWG	3	0.4795927	0.4658424	0.4516602	$q_1 \succ q_2 \succ q_3$
	4	0.4703139	0.4618056	0.4486142	$q_1 \succ q_2 \succ q_3$
	5	0.4630513	0.4581448	0.4460911	$q_1 \succ q_2 \succ q_3$

Fig. 6 Comparison chart for GDIVPHFEWA–GDIVPHFEWG operators with different values of λ

6 Comparative study

To show the feasibleness of the proposed MPDM strategies, the rankings of our proposed operators DIVPHFWA, DIVPHFEWA, DIVPHFWG and DIVPHFEWG are compared with the rankings obtained by the existing MPDM methods as presented in Table 11. The hesitant fuzzy set could be a special case of IVPHFS. For instance, the component $(Y_1, G_1) = (0.2, 0.4, 0.7)$ in Table 1 of Xia and Xu (2011) will be viewed as an IVPHFE as $\{\{(0.2, 0.2), (0.4, 0.4), (0.7, 0.7)\}, \{(0, 0)\}, \{(0, 0)\}\}$. A dual hesitant fuzzy set is a special case of IVPHFS due to the fact the element $(A_1, C_1) =$

Table 11 Comparison of our proposed method (Algorithm 1 and Algorithm 2) with some existing methods

Model	Problem	Existing Method		Proposed method	
		Operator	Ranking Order	Operator	Ranking order
Hesitant Fuzzy Set	Example 4 in Xia and Xu (2011)	$GHFWA_1$ in Xia and Xu (2011)	$A_4 \succ A_2 \succ A_1 \succ A_3$	DIVPHFWA	$A_4 \succ A_2 \succ A_1 \succ A_3$
				DIVPHFEWA	$A_4 \succ A_2 \succ A_1 \succ A_3$
		$GHFWG_2$ in Xia and Xu (2011)	$A_4 \succ A_1 \succ A_3 \succ A_2$	DIVPHFWG	$A_4 \succ A_1 \succ A_3 \succ A_2$
				DIVPHFEWG	$A_4 \succ A_1 \succ A_3 \succ A_2$
Dual Hesitant Fuzzy Set	*Numerical Example in Ju et al. (2014b)	DHFHHA ($\gamma = 1, 2, 5$) in Ju et al. (2014b)	$A_3 \succ A_4 \succ A_2 \succ A_1$	DIVPHFWA	$A_3 \succ A_4 \succ A_2 \succ A_1$
				DIVPHFEWA	$A_3 \succ A_4 \succ A_2 \succ A_1$
		DHFHHG ($\gamma = 1, 2, 5$) in Ju et al. (2014b)	$A_3 \succ A_2 \succ A_4 \succ A_1$	DIVPHFWG	$A_3 \succ A_2 \succ A_4 \succ A_1$
				DIVPHFEWG	$A_3 \succ A_2 \succ A_4 \succ A_1$
Interval-Valued Dual Hesitant Fuzzy Set	Table 1 in Zhang et al. (2014)	IVDHFWA in Zhang et al. (2014)	$A_4 \succ A_3 \succ A_1 \succ A_2$	DIVPHFWA	$A_4 \succ A_3 \succ A_1 \succ A_2$
				DIVPHFEWA	$A_4 \succ A_3 \succ A_1 \succ A_2$
		IVDHFEWA in Zhang et al. (2014)	$A_4 \succ A_3 \succ A_1 \succ A_2$	DIVPHFWG	$A_4 \succ A_3 \succ A_1 \succ A_2$
				DIVPHFEWG	$A_4 \succ A_3 \succ A_1 \succ A_2$
Picture Fuzzy Set	Illustrative Example in Garg (2017)	PFWA in Garg (2017)	$A_2 \succ A_3 \succ A_1 \succ A_4$	DIVPHFWA	$A_2 \succ A_3 \succ A_1 \succ A_4$
				DIVPHFEWA	$A_2 \succ A_3 \succ A_1 \succ A_4$
				DIVPHFWG	$A_2 \succ A_3 \succ A_1 \succ A_4$
				DIVPHFEWG	$A_2 \succ A_3 \succ A_1 \succ A_4$

Table 11 continued

Model	Problem	Existing Method		Proposed method	
		Operator	Ranking Order	Operator	Ranking order
Picture Hesitant Fuzzy Set	Table 2 in Wang and Li (2018)	GPHFWA ($\lambda = 1$) in Wang and Li (2018)	$A_3 \succ A_2 \succ A_5 \succ A_1 \succ A_4$	DIVPHFWA	$A_3 \succ A_2 \succ A_5 \succ A_1 \succ A_4$
				DIVPHFEWA	$A_3 \succ A_2 \succ A_5 \succ A_1 \succ A_4$
		GPHFWG ($\lambda = 1$) in Wang and Li (2018)	$A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$	DIVPHFWG	$A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$
				DIVPHFEWG	$A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$

{{0.3, 0.4}, {0.6}} in Table 1 of Ju et al. (2014b) can be converted into IVPHFE as {{(0.3, 0.3), (0.4, 0.4)}, {(0, 0)}, {(0.6, 0.6)}}. The interval-valued dual hesitant fuzzy element $(A_1, C_1) = \{\{[0.3, 0.4], [0.4, 0.5], [0.5, 0.6]\}, \{[0.2, 0.3], [0.2, 0.4]\}\}$ in Table 1 of Zhang et al. (2014) is transformed to IVPHFE as {{[0.3, 0.4], [0.4, 0.5], [0.5, 0.6]}, {[0, 0]}, {[0.2, 0.3], [0.2, 0.4]}}. The picture fuzzy element $(A_1, G_1) = (0.2, 0.1, 0.6,)$ in illustrative Example of Garg (2017) is modified to IVPHFE as {{(0.2, 0.2)}, {(0.1, 0.1)}, {(0.6, 0.6)}}. Moreover, Picture hesitant fuzzy element $(A_1, C_1) = \{\{0.4, 0.5, 0.7\}, \{0.05\}, \{0.1, 0.2\}\}$ in Table 2 of Wang and Li (2018) also can be modified to IVPHFE as {{(0.4, 0.4), (0.5, 0.5), (0.7, 0.7)}, {(0.05, 0.05)}, {(0.1, 0.1), (0.2, 0.2)}}. It leads to comparing our proposed model IVPHFS with the existing models HFS, DHFS, IVDHFS, PFS, and PHFS. Our proposed operators DIVPHFWA, DIVPHFEWA, DIVPHFWG and DIVPHFEWG provide the identical ranking in evaluating with the prevailing operators GHFWA$_1$, GHFWA$_2$ Xia and Xu (2011), DHFHHA, DHFHHG ($\gamma = 1, 2, 5$) Ju et al. (2014b), IVDHFWA, IVDHFEWA Zhang et al. (2014), PFWA Garg (2017), GPHFWA, GPHFWA ($\lambda = 1$) Wang and Li (2018) of the models HFS, DHFS, IVDHFS, PFS, and PHFS, respectively. It is proven in Table 11.

7 Conclusion

In this paper, we studied and described the notions and operations of IVPHFSs and IVPH-FEs. Some basic properties of the IVPHFEs were also introduced and investigated. We discussed the MPDM problems under interval-valued picture hesitant fuzzy environment. Then, motivated by the idea of traditional dynamic aggregation operators and Einstein operations, some dynamic interval-valued picture hesitant aggregation operators: DIVPH-FWA operator, DIVPHFWG operator, DIVPHFEWA operator, DIVPHFEWG operator, and their generalized versions were proposed to aggregate the interval-valued picture hesitant fuzzy information collected at different periods. In addition, we proposed these dynamic aggregation operators to develop new approaches to deal with the interval-valued picture hesitant fuzzy multi-period decision-making problems. The practical examples and a comparison study were presented to verify the practicality and effectiveness of newly proposed approaches.

The IVPHFS introduced in this paper is quite an extensive version of many (extended) fuzzy sets, and hence the proposed aggregation operators for IVPHFVs are more inclusive than those of many fuzzy variables. In addition, the novelty of the interval-valued picture hesitant fuzzy aggregation operators proposed in this paper is that they can realize the synthesis of the interval-valued picture hesitant fuzzy information collected at different periods, so they develop and enrich the theories of dynamic aggregation operators for hesitant fuzzy elements, picture fuzzy elements, picture hesitant fuzzy elements.

In the future, we will extend the theory of aggregation operators for IVPHFVs and investigate information measures to deal with the decision-making problems in medical diagnosis and pattern recognition under interval-valued picture hesitant fuzzy environment.

Declarations

References

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