Fractional Interaction of Financial Agents in a Stock Market Network

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Abstract

In this study, we present a model which represents the interaction of financial companies in their network. Since the long time series have a global memory effect, we present our model in the terms of fractional integro-differential equations. This model characterize the behavior of the complex network where vertices are the financial companies operating in $XU100$ and edges are formed by distance based on Pearson correlation coefficient. This behavior can be seen as the financial interactions of the agents. Hence, we first cluster the complex network in the terms of high modularity of the edges. Then, we give a system of fractional integro-differential equation model with two parameters. First parameter defines the strength of the connection of agents to their cluster. Hence, to estimate this parameter we use vibrational potential of each agent in their cluster. The second parameter in our model defines how much agents in a cluster affect each other. Therefore, we use the disparity measure of PMFGs of each cluster to estimate second parameter. To solve model numerically we use an efficient algorithmic decomposition method and concluded that those solutions are consistent with real world data. The model and the solutions we present with fractional derivative show that the real data of Borsa Istanbul Stock Exchange Market always seek for an equilibrium state.

Keywords: Network Modelling, Stock Market Network, Fractional Calculus, Caputo Fractional Derivative.
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1 Introduction

Complex systems are mathematical structures involving interacting agents at different levels. These interactions emerge from the financial, chemical, social, and computer system entities. In the realm of computational finance, a financial market can be viewed as interacting group of boundedly-rational agents and its fluctuation represent strong nonlinearity and persistent memory. The mathematical tools such as network and graph theories can be used to understand and analyze these systems [1, 2]. There are several models expressed in the terms of differential equations in biological complex systems. For instance, the virus models that classify individuals and

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hosts can be used to analyze spread of a contagion [3–5]. Besides, bursting electrical activity in the pancreatic β-cell, population models, and unilingual-bilingual interactions [6], and the interaction of biological species living together [7] can also be modelled by differential equations. However, these models are not only restricted to biological systems. Recent studies show that complex systems involving financial agents have similar structures as systems involving biological agents [8, 9]. Therefore, it is reasonable to model the interaction of financial agents as we model the interaction of biological agents.

In a financial market, heterogeneous agents interact through simple investment strategies driven by the investors. In a perfect rational market, information is transmitted continuously and agents adopt their behavior accordingly. Besides, asset prices reflect economic fundamentals. Agents are considered as they only interact though price system. Hence, a complex network where agents are expressed as vertices and edges are formed by the correlation of price fluctuations emerges as a powerful mathematical tool to model such a financial system in the traditional way. In contrast to Keynesian approach, such traditional way takes account of prices of assets as they are only driven by market fundamentals and the role of market psychology is neglected. Even though we use the traditional way to express our model in this study, we need to point out two important classes of investors which are called chartists and fundamentalists in the traditional way of interaction of agents [10]. Chartists tend to look for simple patterns such as trends, past prices, and base and make their investment upon those patterns. Conversely, fundamentalists make their decisions upon the expectation of asset price as moving towards its fundamental value. The fundamentalist investors buy or sell assets that are under or overvalued. The market tends to be dominated by one of fundamentalists or chartists. However, since the behavior of the agents is persistent, the majority of agents switches to the other view at certain point [11, 12].

Our approach in this study aims to model interaction of agents in a stock market network in the traditional way. We first use a threshold method to construct a network model where vertices are the companies operating in a stock market and edges are formed by the correlation distance of daily logarithmic returns of stock prices. The dimensionality of the resulting network model would be really high and the patterns that yield power law of degree distributions would be disappeared, however it would involve optimally many edges to characterize community structures. By maximization of the modularity of edges in the network, we can cluster agents into densely connected vertex sets. Then, each cluster has its subdominant ultra-metric structure that is a hierarchical structure with at least one leading actor. We set the number of cluster to two, then assume the investors, even chartists or fundamentalists, start to invest one cluster regarding to factors such that merging, capital augmentation, public flotation, etc. Then, the investors in the other cluster start to sell assets to get the capital to invest increasing valued assets. Therefore, the price fluctuation spread within each cluster by conducting leading actors. However, at certain time, the profit realizations start within the asset price increasing cluster, and then the capital emergent by the profit realization is used to invest assets in price decreasing cluster. Eventually, the two clusters find an equilibrium state.

In the complex network model of financial agents the interactions are modelled by the correlations of long time series [13–19]. Beside the other types of complex systems, financial systems have the strong memory and heredity properties. Therefore, while using differential equations in financial models it is much more useful to get fractional calculus involved. Fractional calculus is the extension of the integer order differential and integral operators to fractional orders [20, 21]. The dynamic memory in a financial process can be defined as the averaged characteristic that describes the dependence of a process in the past. Such memory assumes withitness of financial agents about the history of the process. In formal way, the information on the state of the process $\{t, \chi(t)\}$ does not only affect the behavior of financial agents, but also the information about the process state $\{\tau, \chi(\tau)\}$ also has effect at $\tau \in [0,t]$. This effect is related with the fact that the change of the factors can leads to different amount of change in indicators that is there exist multivalent dependencies among variables. One type of such memory of financial agents is called the fading memory and have range application area in physical sciences [22–30]. In this study, we assume that financial agents can remember the previous changes of investments and the impact of these changes on the output by following fading memory by using Caputo's definition of fractional derivative.

In this study, we present our model with fractional derivative as similar as the model describing biological species living together. This model of biological species is first given in [7] as the following usual integro-differential equations:

$$\frac{d\mu_1}{dt} = \mu_1(t) \left[k_1 - \gamma_1 \mu_2(t) - \int_{t-T_0}^{t} f_1(t-\tau)\mu_2(\tau)d\tau \right], \;\; k_1 > 0 \tag{1}$$

$$\frac{d\mu_2}{dt} = \mu_2(t) \left[-k_2 + \gamma_2 \mu_1(t) + \int_{t-T_0}^{t} f_2(t-\tau)\mu_1(\tau)d\tau \right], \;\; k_2 > 0. \tag{2}$$

Several solution methods are also presented to study this model [31–33]. The characterization of the fractional order of the model is also studied in [34].

The rest of the paper is organized as follow: In Section 2, we present the preliminaries about to fractional calculus and graph theoretical concepts that we use throughout the paper. We start our analysis by first determining the financial agents in Section 3. The stock market we choose to study is Borsa Istanbul Stock Exchange Market ($XU100$). The agents are the companies operating in $XU100$ and expressed with the time series of the time span of working days from 2013 to 2015. Afterwards, we determine the clusters of financial agents that have strong correlations by using high modularity method. Afterwards we introduce the Adomian decomposition method for solving the system numerically. In Section 5, we present the results by solving the model with Adomian decomposition method. And finally, in Section 6, we deeply discuss the computational results.

2 Preliminaries

Fractional calculus is an efficient mathematical tool to express complex system phenomenon which involve memory effect. Hence, we use fractional derivatives and integrals to study the model we present in this study. In this section, we give some basics about the fractional calculus in Caputo sense. We also introduce some basics about the graph theory which is the fundamental tool for network modelling. Throughout the paper we let Γ to represent Gamma function that is an extension of the factorial function.

2.1 Fractional Calculus

The generalization of the integer order differentiation and integration to the fractional order is called fractional calculus [20, 21]. The basic definitions and properties of fractional calculus theory is given as follows:

Definition 1. [20] For $f(x) \in \mathscr{C}(a,b)$ and $n-1 < \alpha \leq n$, the Caputo fractional derivative operator of order α is given as

$$^{C}_{a}D^{\alpha}_{t}f(t) = \frac{1}{\Gamma(n-1)} \int_{a}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} \, d\tau.$$

Throughout this paper, we denote the Caputo fractional derivative operator as $^{C}_{a}D^{\alpha}_{t} = D^{\alpha}_{a}$. We also let $a = 0$ since our formulation only involves the initial conditions as $t = 0$.

Definition 2. [20] The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function f is defined as

$$J^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (x-\tau)^{\alpha-1}f(\tau) \, d\tau, \;\; \alpha > 0, t > 0$$

$$J^{0}f(t) = f(t).$$

The several properties of the Riemann-Liouville fractional integral operator can be found in [35–37]. Since the Caputo fractional derivative allows traditional initial and boundary conditions to be included in the formulation of the problem [38], we present our model in the sense of Caputo fractional derivative. By the introduction

of the J^α, the D_a^α can also be expressed as

$$D_0^\alpha f(t) = J^{m-\alpha} D^m f(t),$$

where $m - 1 < \alpha \le m$, $m \in \mathbb{N}$, $t > 0$.

Also, the following two basic properties of the entwined relations among Caputo and Riemann-Liouville fractional operators are needed to present the solution of the fractional differential equations.

Lemma 1. *[35] If $m - 1 < \alpha \ge m$, $m \in \mathbb{N}$, then*

$$D_0^\alpha J^\alpha f(t) = f(t)$$

and

$$J^\alpha D_0^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}, \quad t > 0.$$

2.2 Graph Theory

The real–world problems are often expressed with the relations of interacting individuals. One of the efficient mathematical tools to represent such relations is the simple graphs. In the Stock Market Networks, interactions of financial agents can be modelled by simple graphs. Let V be the set of the interacting individuals and E be the set of relations, then a simple graph is a tuple $G = (V, E)$. Here we call V as the vertex set and E as the set of edges. The each element of E is the unordered pair of vertices as $e_k = (v_i, v_j)$, where $v_i, v_j \in V$ for all $i, j, k \in \mathbb{N}$. The number of edges incident to a vertex v is called as the degree of v and we denote the degree as d_v.

A sequence of edges between the vertices v_i and v_j is called a path, and if there is a path between any vertices of the graph G, then G is called as connected. If there is an edge between all elements of V, then G is called as a complete graph. The k-clique of the graph G is the complete subgraph of G which involves k-many vertices of G.

For the simple graph $G = (V, E)$ with the unordered edges, a binary matrix which has the entities as

$$A_G(i, j) = \begin{cases} 1, \text{ if } (v_i, v_j) \in E \\ 0, \text{ otherwise} \end{cases}$$

is used to represent the relations and called as adjacency matrix. A_G is symmetric by definition.

Now let $|V| = n$ and D_G be the diagonal degree matrix of G defined as $D_G = diag[d_{v_1}, \ldots, d_{v_n}]$. The matrix $L_G = A_G - D_G$ is called the Laplacian matrix of G. The spectrum of the L_G encodes structural properties of G. The one that we use in this study is helpful to construct a threshold network of the financial agents. All eigenvalues of L_G are positive semi-definite with the least one 0. The multiplicity of the 0 eigenvalue equals the connected components number of G [39].

Several types of subgraphs also involve the information about the network which is expressed as a simple graph $G = (V, E)$. One of them is the tree structure that has minimum weight. Such subgraphs are called as Minimum Spanning Tree (MST) and involve the junction vertices which are dominant in the flow of information and comes up with subdominant ultra-metric structure [40, 41]. In the case of financial agents are the vertices of the network, MST gives the hierarchical structure of the financial network [42–44]. A planar graph is a simple graph that can be embedded on the plane, which is none of the graph edges intersect. Trees like planar graph that involve cliques are also useful to extract information about the network. Such tree like planar graphs have the same hierarchical structure as MST but they contain larger amount of information about the relation among the interacting agents [45–47]. In [45], authors present a method to obtain a planar graph with maximum non-crossing edges among the agents of a network and called it Planar Maximally Filtered Graph (PMFG).

3 Model

This study involves 93 companies that have been operating in Borsa Istanbul 100 Stock Exchange Index ($XU100$) from January 2013 to January 2015. The Pearson correlation coefficient of time series assumes the equality of the length of time series. Hence, even though $XU100$ has 100 operating companies, we only consider 93 of them which have the equal time length. Trading hours for the stocks are held by two sessions on business days with mid-day break, and one session in some official holidays [44, 48]. The tickers of the companies operating in $XU100$ and that are considered in this study is given in Table 1. The more details on the data can be found in [44].

Financials
AKBNK, SKBNK, SNGYO, TSKB, TEKST, TRGYO, VKGYO, ALGYO, ISGYO, GARAN, ALBRK, GLYHO, ISCTR, YKBNK, SAHOL, GOZDE, HALKB, VAKBN, ECZYT, SAFGY, EKGYO, SAHOL, GSDHO
Industrials
ASELS, TAVHL, TKFEN, TTRAK, CLEBI
Consumer Discretionary
ASUZU, TKNSA, TOASO, YAZIC, AKSA, ARCLK, GSRAY, KARSN, THYAO, BRISA, DOAS, FENER, MNDRS, METRO, VESBE, ADEL , BJKAS, NTTUR, GOODY, OTKAR, TMSN, EGEEN, FROTO, IHLAS
Energy
AYGAZ, TUPRS, IPEKE , KCHOL
Technology
NETAS, VESTL
Materials
SASA, AFYON, ANACM, BAGFS, CIMSA, KONYA, KOZAA, ERBOS, KRDMD, PRKME, SISE, ALKIM , TRKCM, GUBRF, KOZAL, BRSAN, KARTN, PETKM, GOLTS, EREGL
Communications
TTKOM, TCELL, DOHOL, HURGZ
Consumer Staples
AEFES, CCOLA, BIZIM, ECILC, BIMAS, MGROS, SODA, ULKER
Utilities
AKSEN, ALARK, TRCAS, ZOREN, ENKAI

Table 1 Sectors of each considered stock

The data we use is available with sessional closure price, therefore we calculate sessional closure price logarithmic return as

$$Cl_i = \log P_i(t+1) - \log P_i(t),$$

where $P_i(t)$ is the closure price of the i-th stock at the session t. To represent the relation between stock pairs, we use the Pearson correlation coefficient of stocks as

$$\rho_{ij} = \frac{<Cl_i Cl_j> - <Cl_i><Cl_j>}{\sqrt{(<Cl_i^2> - <Cl_i>^2)(<Cl_j^2> - <Cl_j>^2)}},$$

where $< \cdots >$ is the temporal average performed on the trading days.

Pearson correlation coefficient varies between -1 and 1. $\rho_{ij} = 1$ indicates that the stocks i and j are completely correlated whilst ρ_{ij} indicates that the stocks i and j are completely uncorrelated [42]. Hence, it is also possible to introduce a new distance $d_{Corr} := \sqrt{2(1 - \rho_{ij})}/2$ as in [13,44]. We can conclude that if $d_{Corr}(i, j) = 0$, then the stocks i and j are completely correlated, and if $d_{Corr}(i, j) = 1$, then the stocks i and j are completely uncorrelated.

This distances based on Pearson correlation is helpful to us for edge determination on the network. By using an empirical threshold value among the stocks, it is possible to determine edges representing strong relations as

$$A_G(i, j) = 1 \;\; iff \;\; d_{Corr}(i, j) \leq ThV,$$

where ThV is the threshold value. The threshold value can be determined by the subdivision of the interval $[0, 1]$, where the boundaries are the extremal values of d_{Corr}, into h many subintervals. The details on the algorithm of network construction and computational complexities can be found in [13,44].

The model we proposed in this study first deal with the two clusters of financial agents of the network. In the literature, the cluster of densely connected vertices of a network is called graph communities [49,50]. This densely connection is internal and can be used to analyze the relations that are represented by edges on the network. There are several methods to detect communities in a network [51–54]. To find the graph communities in the network; we use the Modularity Maximization Method which is based on the maximizing the Newman modularity index [51] defined as

$$Q = \sum_{i=1}^{N_C} \left[\frac{E_k}{m} - \frac{1}{4m^2} \left(\sum_{j \in V_k} d_j \right)^2 \right],$$

where E_k is the number of edges in the k-th module, N_C is the total number of modules, m is the total number of edges and d_j is the vertex degree. Since the resulting communities are non-overlapping and this method let us to determine final number of the communities, we choose it as an efficient tool.

Now let us consider the two communities of financial agents with the total number of investments $\mu_1(t)$ and $\mu_2(t)$, respectively, at time t. Let us assume the investment in the first community is increasing with the coefficient of increase k_1 and in the second community is decreasing with the coefficient of decrease k_2. Both coefficients k_1 and k_2 are positive reals. If the two communities are left separate; i.e. they are non-overlapping, then the fractional growth of the first can be represented by

$$D_0^\alpha \mu_1(t) = k_1 \mu_1(t) \tag{3}$$

and the decline of the second community can be represented by

$$D_0^\alpha \mu_2(t) = -k_2 \mu_2(t). \tag{4}$$

The neoclassical liberal economy states that markets always look for the equilibrium state. Hence, if we put these two communities together in the corresponding stock market environment, the decrease of the rate of the increase of first community is proportional to $\mu_2(t)$ and vice versa. Therefore, it is reasonable to assume the increase and decrease coefficient as

$$k_1' = k_1 - \gamma_1 \mu_2(t) \tag{5}$$

and

$$k_2' = k_2 + \gamma_2 \mu_1(t), \tag{6}$$

where γ_1 and γ_2 are the proportionality constants which depend on other investor behavior, respectively. The actual decrease and increase of the investments in the communities are due not only to the presence of the other community but also to all previous presences for the whole time interval $t - T_0 < \tau < t$, where T_0 is the finite heredity duration of both communities. In addition to the present γ_1 and γ_2 factors, we may have the record of

decrease as $f_1(\tau)$ and increase as $f_2(\tau)$. Therefore, by considering the heredity duration of both communities, the total decrease in k_1 in the time interval T_0 is

$$\delta k_1 = -\int_{t-T_0}^{t} f_1(t-s)\mu_2(s)\,ds \tag{7}$$

the total fractional increase in k_2 is

$$\delta k_2 = \int_{t-T_0}^{t} f_2(t-s)\mu_1(s)\,ds. \tag{8}$$

Now, by considering effective values of the k_1 and k_2 and the equations 3–8, the fractional model with fading memory of the equilibrium state of the two communities of financial agents in same stock market can be given as the system of fractional integro-differential equations as follows:

$$D_0^\alpha \mu_1(t) = \mu_1(t)\left[k_1 - \gamma_1\mu_2(t) - \int_{t-T_0}^{t} f_1(t-s)\mu_2(s)\,ds\right], \tag{9}$$

$$D_0^\alpha \mu_2(t) = \mu_2(t)\left[-k_2 + \gamma_2\mu_1(t) + \int_{t-T_0}^{t} f_2(t-s)\mu_1(s)\,ds\right], \tag{10}$$

$$\mu_1(0) = N_1, \quad \mu_2(0) = N_2, \tag{11}$$

where N_1 and N_2 are the initial conditions.

4 Method

In this section, we present the graph theoretical methods to determine parameters k_1, k_2, γ_1, and γ_2 of the fractional integro-differential equation model in the initial value problem 9–11 and the numerical solution that is based on Adomian Decomposition Method.

4.1 Parameter Estimation

The parameters in network models can be estimated by using graph theoretical concepts. For the system 9–11, to determine the coefficients of increase and decrease, we use an interpretation of the displacement of a vertex in a network from its equilibrium state while the network is under a thermal bath. This thermal bath can be seen as the change of investment strategies on given network. This procedure is called vibrational potential and first presented in [55]. The later studies consider vibrational potential as an efficient measure to vertex centrality [56–58]. The main idea to compute the vibrational potential of a network is embedding vertices to n-dimensional Euclidean space by using the Moore-Penrose pseudo-inverse of the Laplacian, where $n = |V|$. Within the hierarchical structure of each community, each stock market tends to be adjacent to junction vertices. Therefore, the change of investment on the junction vertices directly affect the corresponding leaves. Therefore, we correspond the increase/decrease coefficients with vibrational potential of the network. However, instead of direct computing vibrational potential of the network, we compute vibrational potential of each vertex respect to its neighborhood graph.

For this purpose we present the vertex displacement in vibrational potential of a vertex respect to its neighboring vertices with

$$V(\vec{x}_v) = \frac{k}{2}\vec{x}_v^T L_N \vec{x}_v, \tag{12}$$

where k is the spring constant, L_N is the Laplacian of the neighboring graph G_N of the vertex v in G, and $\vec{x}_v$ is the vector whose i-th entry is the displacement of v. The mean displacement of the vertex v can be computed with the reverse temperature β as

$$\Delta x_i = \sqrt{\int x_i^2 P(\vec{x}_v)d\vec{x}_v}, \tag{13}$$

where the probability distribution $P(\vec{x}_v)$ is

$$P(\vec{x}_v) = \exp\left(-\frac{\beta k}{2}\right) \Big/ \int \exp\left(-\frac{\beta k}{2}\right) d\vec{x}_v. \tag{14}$$

Similarly the displacement correlation of the vertices in same neighborhood can be defined as

$$< x_i, x_j > = \int x_i x_j P(\vec{x}_v) d\vec{x}_v, \tag{15}$$

where $< \cdots >$ is the thermal average.

Let $0 = \lambda_1^N < \lambda_2^N \leq \ldots \leq \lambda_n^N$ be the spectrum of L_N respect to eigenvalues λ_μ^N. Since the quantity respect to 0 eigenvalue is the center of mass, the 0 eigenvalue does not affect the vertex displacement. Then the integral measure can be transformed by

$$d\vec{x}_v = \prod_{i=1}^n dx_i = |\det U_N| \prod_{i=1}^n d\xi_i = d\vec{\xi}_v \tag{16}$$

where U_N is the matrix formed by the orthogonal eigenvectors of L_N. By the introduction of this transform the new probability distribution can be obtained as

$$P(\vec{\xi}_v) = \exp\left(-\frac{\beta k}{2}\vec{\xi}_v^T \Lambda_N \vec{\xi}_v\right) \Big/ \int \exp\left(-\frac{\beta k}{2}\vec{\xi}_v^T \Lambda_N \vec{\xi}_v\right) d\vec{\xi}_v$$
$$= \exp\left(-\frac{\beta k}{2}\vec{\xi}_v^T \Lambda_N \vec{\xi}_v\right) \Big/ \prod_{\mu=1}^n \int_{-\infty}^\infty \exp\left(-\frac{\beta k}{2}\lambda_\mu^N \xi_\mu^2\right), \tag{17}$$

where the diagonal matrix Λ_N involves the eigenvalues λ_μ^N.

Since 0 eigenvalue does not effect the vertex displacement, we can remove the component $\mu = 1$ from the Equation 17, and the probability distribution can be computed as

$$P(\vec{\xi}_v) = \exp\left(-\frac{\beta k}{2}\vec{\xi}_v^T \Lambda_N \vec{\xi}_v\right) \Big/ \prod_{\mu=2}^n \int_{-\infty}^\infty \exp\left(-\frac{\beta k}{2}\lambda_\mu^N \xi_\mu^2\right)$$
$$= \exp\left(-\frac{\beta k}{2}\vec{\xi}_v^T \Lambda_N \vec{\xi}_v\right) \Big/ \prod_{\mu=2}^n \sqrt{\frac{2\pi}{\beta k \lambda_\mu}}. \tag{18}$$

Hence, by using the probability distribution obtained in Equation 18, it is possible to compute the Equation 13 as

$$\Delta x_i = \sqrt{< x_i^2 >} = \sqrt{\mathscr{L}_i \Big/ \prod_{\mu=2}^n \sqrt{\frac{2\pi}{\beta k \lambda_\mu}}}, \tag{19}$$

where

$$\mathscr{L}_i = \sum_{j=2}^n \int_{-\infty}^\infty (U_{ij}\xi_j)^2 \exp\left(-\frac{\beta k}{2}\lambda_j \xi_j^2\right) d\xi_j \times \prod_{\mu=2,\mu\neq j}^n \int_{-\infty}^\infty \exp\left(-\frac{\beta k}{2}\lambda_\mu \xi_\mu^2\right) d\xi_\mu$$
$$= \prod_{\mu=2}^n \sqrt{\frac{2\pi}{\beta k \lambda_\mu}} \times \sum_{j=2}^n \frac{U_{ij}^2}{\beta k \lambda_j}.$$

Therefore, the mean displacement of a vertex from its neighborhood can be computed as

$$\Delta x_i = \sqrt{\sum_{j=2}^n \frac{U_{ij}^2}{\beta k \lambda_j}}.$$

By the introduction of the Moore-Penrose pseudo inverse L_{ij}^{N+} of L_N as in [59, 60], it is also possible to compute the mean displacement of a vertex from its neighborhood as

$$\Delta x_i = \frac{1}{\beta k}(L_{ii}^{N+}).\tag{20}$$

We also note that the displacement correlation of the vertices in the same neighborhood that is given in the Equation 15 can be computed in the terms of Moore-Penrose pseudo inverse as

$$<x_i, x_j> = \frac{1}{\beta k}(L_{ij}^{N+}).\tag{21}$$

Another parameters we need to estimate in the system 9–11 are γ_1 and γ_2 which are the proportionality values. The proportionality values control how much financial agent in the same community affect each other. Hence, they can be measured as how strong each agents are connected internally. This measurement is naturally arise from the PMFG of each communities. PMFG structure allows cliques which are the topological subgraph structure representing strong relations. Since PMFG also has the information about the hierarchical structure, it is reasonable to measure internal connectedness of communities by using PMFG. For this measurement we follow the way presented in [45]. The mean disparity measurement $< y >$ of a PMFG can be defined as the mean of

$$y(i) = \sum_{i \neq j, j \in Clique} \left(\frac{d_{Corr}(i, j)}{s_i} \right)^2,\tag{22}$$

where i is the generic element of the clique and

$$s_i = \sum_{i \neq j, j \in Clique} d_{Corr}(i, j).\tag{23}$$

4.2 Adomian Decomposition Method

It is well known that many nonlinear differential equations exhibit strange attractors and their solutions have been discovered to move toward strange attractors. If these strange attractors are examined deeply, it can be seen that these are fractals. Therefore, we aim to deal with fractal nonlinear differential equations rather than classical forms of them. Hence we shall extend the Adomian decomposition method to be used for solving fractional nonlinear equations. For the solution of the system 9–11, we use an efficient decomposition method for approximating the solution of systems of fractional integro-differential equation that are given in Caputo sense. The approximate solutions are calculated in the terms of a convergent series as in [34].

Now let us consider the system 9–11 with $0 < \alpha \leq 1$. By following the decomposition idea we may state that

$$D_0^{\alpha} \mu_1(t) = m_1(t),\tag{24}$$
$$D_0^{\alpha} \mu_2(t) = m_2(t).\tag{25}$$

This equations lead us to the integral equations

$$\mu_1(t) = \mu_1(0) + J^\alpha(m_1(t)) \tag{26}$$

$$= N_1 + J_0^\alpha m_1(s)ds, \tag{27}$$

$$\mu_2(t) = \mu_2(0) + J^\alpha(m_2(t)) \tag{28}$$

$$= N_2 + J_0^\alpha m_2(s)ds, \tag{29}$$

$$m_1(t) = \mu_1(t)\left(k_1 - \gamma_1\mu_2(t) - \int_{t-T_0}^t f_1(t-s)\mu_2(s)ds\right) \tag{30}$$

$$= k_1\left(N_1 + \int_0^t m_1(s)ds\right) - \mu_1(t)\left(\gamma_1\mu_2(t) - \int_{t-T_0}^t f_1(t-s)\mu_2(s)ds\right),$$

$$m_2(t) = \mu_2(t)\left(-k_2 + \gamma_2\mu_1(t) - \int_{t-T_0}^t f_2(t-s)\mu_1(s)ds\right) \tag{31}$$

$$= k_2\left(N_2 + \int_0^t m_2(s)ds\right) + \mu_2(t)\left(\gamma_2\mu_1(t) + \int_{t-T_0}^t f_2(t-s)\mu_1(s)ds\right).$$

Afterwards, the Adomian process will be as follows:

$$\mu_{1,0} = N_1, \ \mu_{2,0} = N_2, \tag{32}$$

$$m_{1,0} = k_1 N_1, \ m_{2,0} = k_2 N_2, \tag{33}$$

$$\mu_{1,j+1} = J_0^\alpha m_{1,j}(s)ds, \ \mu_{2,j+1} = J_0^\alpha m_{2,j}(s)ds, \tag{34}$$

$$m_{1,j+1} = k_1\int_0^t m_{1,j}(s)ds - \gamma_1\sum_{k=0}^j \mu_{1,k}(t)\mu_{2,j-k}(t) \tag{35}$$

$$- \int_{t-T_0}^t f_1(t-s)\left(\sum_{k=0}^j \mu_{1,k}(t)\mu_{2,j-k}(t)\right)ds$$

$$m_{2,j+1} = k_2\int_0^t m_{2,j}(s)ds - \gamma_2\sum_{k=0}^j \mu_{1,k}(t)\mu_{2,j-k}(t) \tag{36}$$

$$+ \int_{t-T_0}^t f_2(t-s)\left(\sum_{k=0}^j \mu_{1,k}(t)\mu_{2,j-k}(t)\right)ds.$$

5 Results

In order to study the proposed model in the Borsa Istanbul Stock Exchange, we first apply our algorithm to the data set to obtain stock market network. For the fraction size $h = 10000$, the algorithm determines the control parameter as 0.6854. The vertices are sorted from 1 to 93 by the alphabetical order in Table 1. The formed network is presented in Figure 1. The vertices with maximum vertex degree are ADEL, AKBNK, AKSEN, ALBRK, ALGYO, ALKIM, ARCLK, ASELS, BRISA, DOAS, GARAN, GOLTS, HALKB, ISCTR, KARTN, KCHOL, KONYA, KRDMD, MGROS, OTKAR, PRKME, SAHOL, SISE, SNGYO, TKFEN, TKNSA, TMSN, TOASO, TRCAS, VAKBN, and YKBNK with the degree number 92, and the vertex with minimum vertex degree is VKGYO. The maximum and mean correlation distances among the companies are 0.7211 and 0.5798, respectively. Hence it can also be concluded this network has strong internal connectedness. The correlation distance matrix of the agents are presented in Figure 2 with temperature mapping.

To determine the two non-overlapping clusters of financial actors we use the high modularity method. The resulting communities are presented in Figure 3. The agent numbers of each community give us the initial conditions as $N_1 = 66$ and $N = 27$.
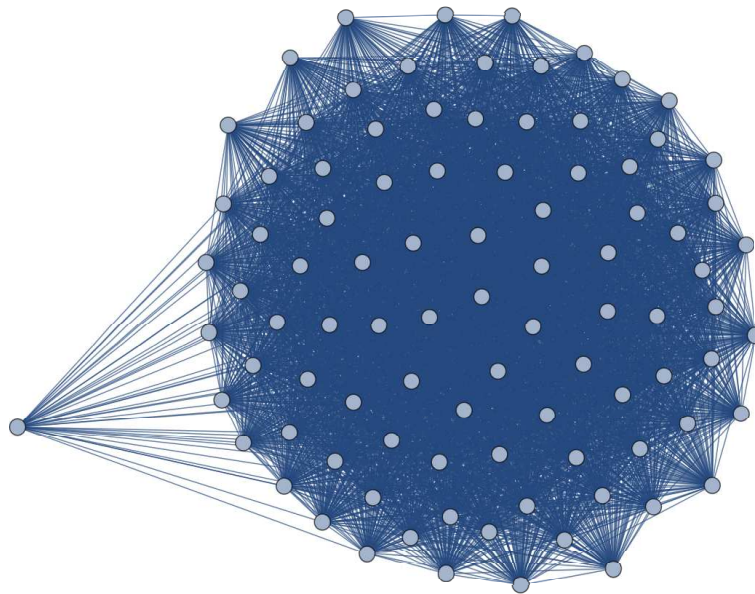
Fig. 1 The network of $XU100$ with the threshold value 0.6854.
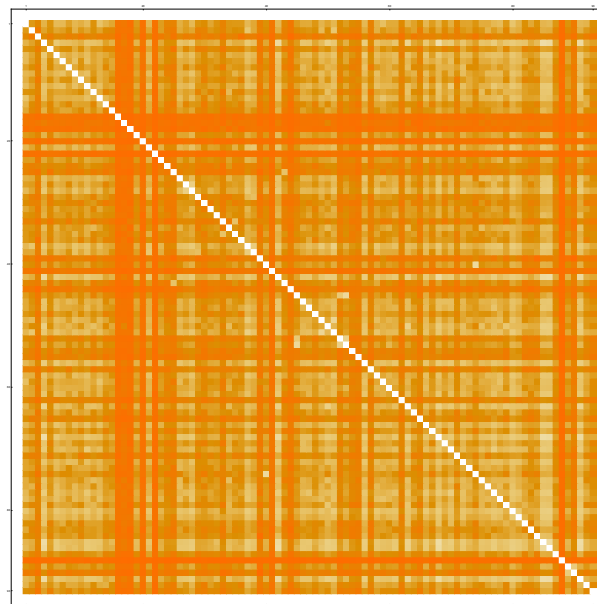
Fig. 2 The matrix of correlation distance among the financial agents of $XU100$. The darker points are closer to 1 while the lighter ones are closer to 0.

As aforementioned, parameters of the model described by the system 9–11 are obtained by vibrational potential respect to neighborhood graphs and mean disparity measures of each communities. We need to note that vibrational potentials of each vertices are tend to form internal clusters; i.e., some of them have higher values and some of them have lower values. Therefore, while determining k_1 and k_2 values, we choose the mean value of each vibrational potentials. Afterwards forming the PMFG of each community, it becomes possible to obtain disparity measures respect to 4-cliques, which are the representation of the strongest connections. The resulting parameters $\gamma_1 = 0.3342$ and $\gamma_2 = 0.3388$. The parameters are close to the value $1/3$ which also states that the clustering method we choose is reasonable [45]. To interpret the results, we present MSTs and PMFGs of both two clusters in Figures 4–5
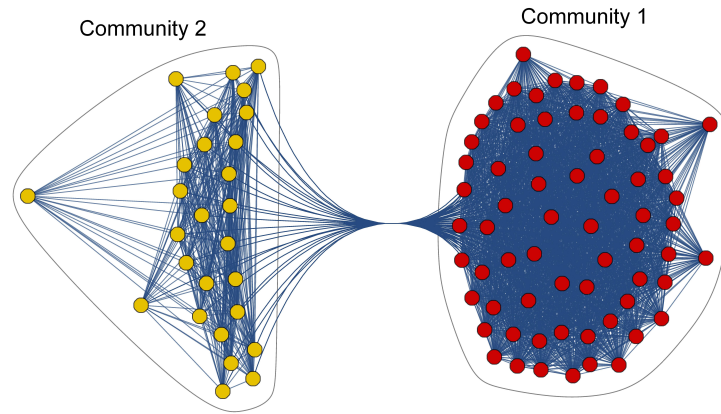
Fig. 3 The community plot of the resulting network.

In the light of these computed parameters we may now state the system 9–11 with T_0=100, $f_1(t) = f_2(t) = e^t$ as

$$D_0^\alpha \mu_1(t) = \mu_1(t) \left[5.95 - 0.3342\mu_2(t) - \int_{t-100}^{t} e^{(t-s)} \mu_2(s)\, ds \right], \tag{37}$$

$$D_0^\alpha \mu_2(t) = \mu_2(t) \left[-4.39 + 0.3388\mu_1(t) + \int_{t-100}^{t} e^{(t-s)} \mu_1(s)\, ds \right], \tag{38}$$

$$\mu_1(0) = 66, \quad \mu_2(0) = 27. \tag{39}$$

By applying the Adomian process obtained in Section 4.2, the solution of the initial value problem 37–39 can be obtained as

$$\mu_1(t) = N_1 + \frac{k_1 N_1 t^\alpha}{\Gamma(1+\alpha)} + \frac{N_1 t^\alpha \left(\left(-1 + e^{-T_0}\right) N_2 + 4^{-\alpha} k_1^2 \sqrt{\pi} t^\alpha / \Gamma(0.5 + \alpha) - N_2 \gamma_1 \right)}{\Gamma(1+\alpha)}, \tag{40}$$

$$\mu_2(t) = N_2 + \frac{k_2 N_2 t^\alpha}{\Gamma(1+\alpha)} + \frac{N_2 t^\alpha \left(\left(1 - e^{-T_0}\right) N_1 + 4^{-\alpha} k_2^2 \sqrt{\pi} t^\alpha / \Gamma(0.5 + \alpha) - N_1 \gamma_2 \right)}{\Gamma(1+\alpha)}. \tag{41}$$
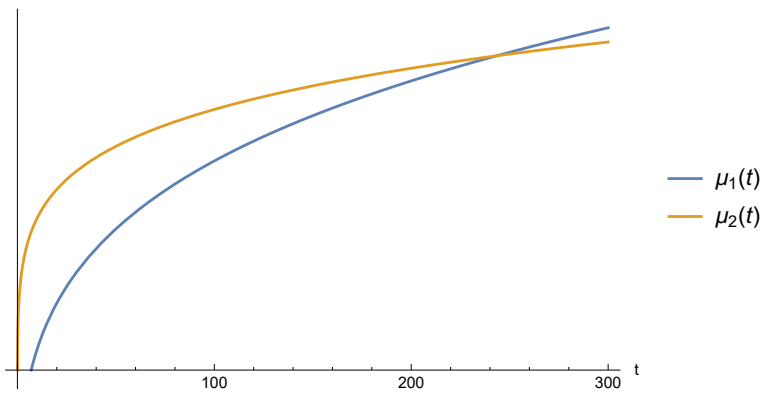
with a three-term approximation.

The plots of the solution functions (40–41) are presented in Figures 6–15 for the different α values.

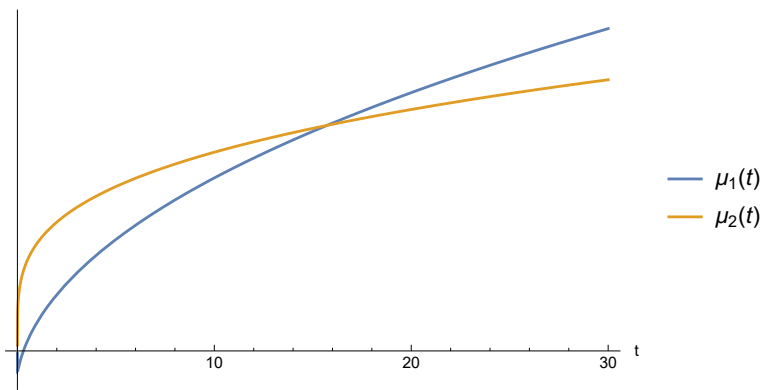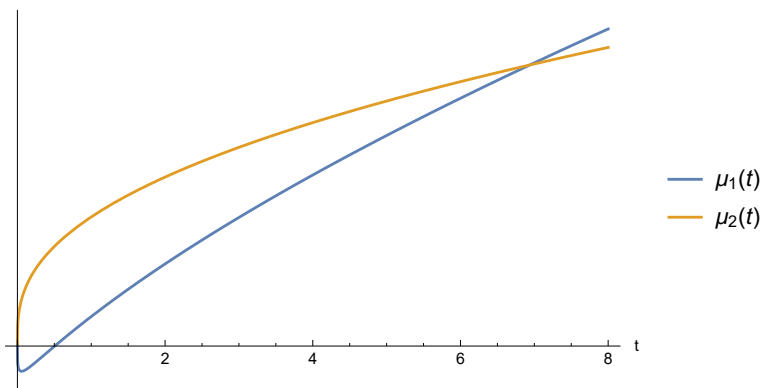**Fig. 4** MST (above) and PMFG (below) filtering of Community 1.
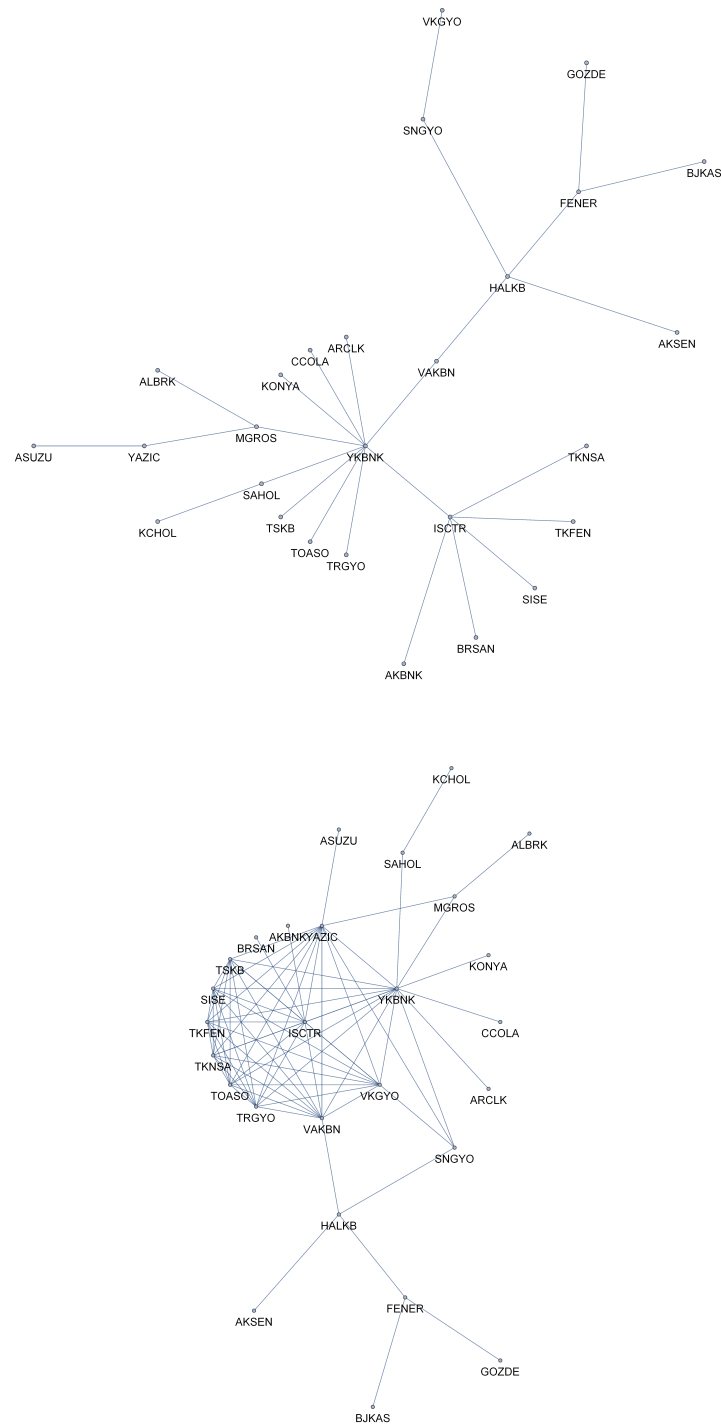
Fig. 5 MST (above) and PMFG (below) filtering of Community 2.

Fig. 6 The solutions of the initial value problem 40–41 for $\alpha = 0.1$
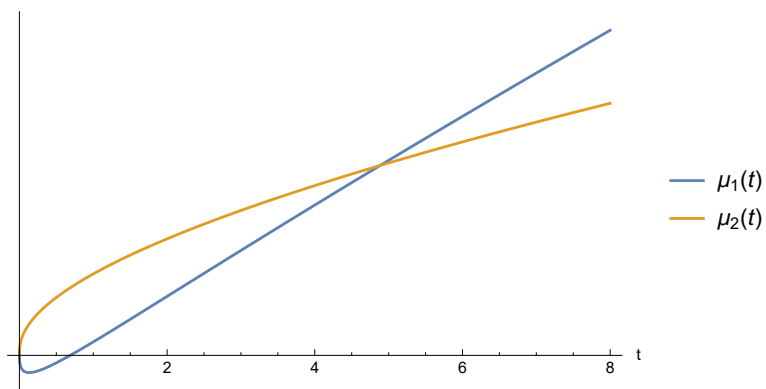

Fig. 7 The solutions of the initial value problem 40–41 for $\alpha = 0.2$
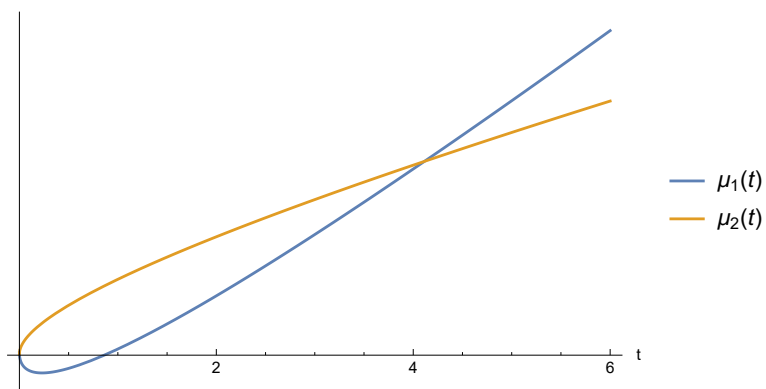

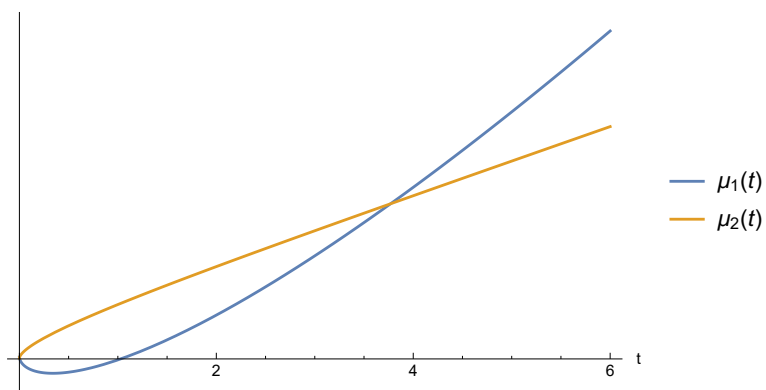**Fig. 8** The solutions of the initial value problem 40–41 for $\alpha = 0.3$
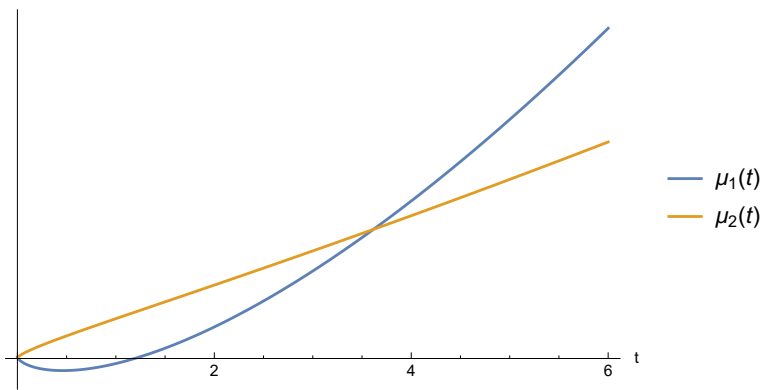
Fig. 9 The solutions of the initial value problem 40–41 for $\alpha = 0.4$

Fig. 10 The solutions of the initial value problem 40–41 for $\alpha = 0.5$

Fig. 11 The solutions of the initial value problem 40–41 for $\alpha = 0.6$
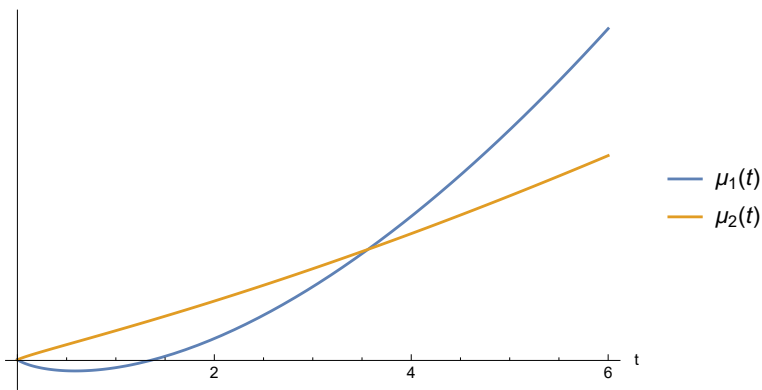
Fig. 12 The solutions of the initial value problem 40–41 for $\alpha = 0.7$

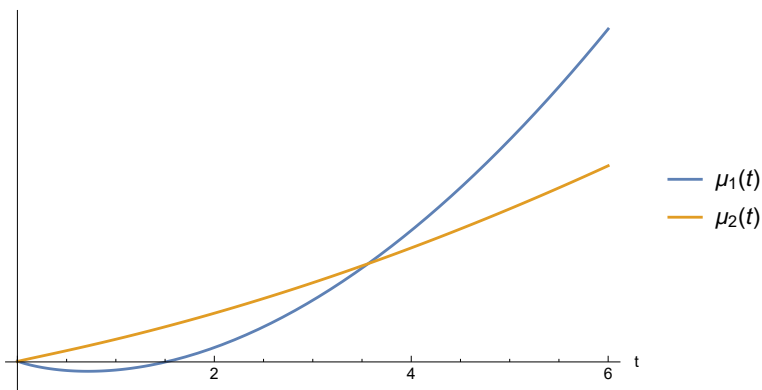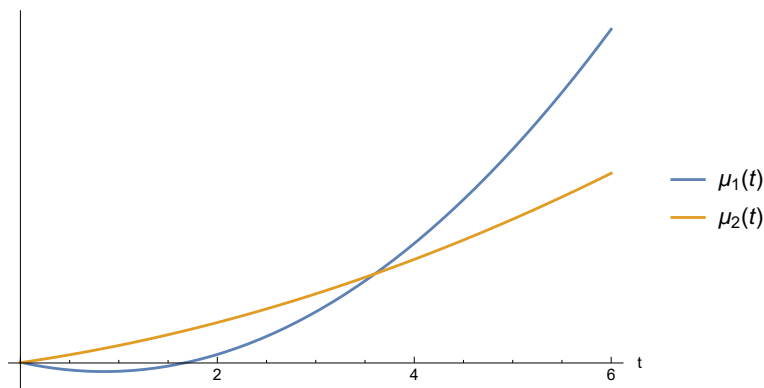**Fig. 13** The solutions of the initial value problem 40–41 for $\alpha = 0.8$

Fig. 14 The solutions of the initial value problem 40–41 for $\alpha = 0.9$

Fig. 15 The solutions of the initial value problem 40–41 for $\alpha = 0.10$

6 Conclusions

Ordinary differential equations are the most common mathematical tool to represent real world problems. But ordinary differential equations become less effective whenever the problem involves memory effect. Complex systems that representing financial agents have the memory effect, hence it is reasonable to model such systems by using the idea of fractional differential.

In this study, we propose a model which is represents the fractional interaction of financial agents. The interaction of the agents is determined within a complex network of a stock market. We express the model as a system of fractional integro-differential equations in Caputo sense. Hence, we keep the fading memory of the financial interaction. Our model considers two clusters of agents where one cluster tends to get investment flow. To determine the clusters we use maximization of the edge modularity in stock market network. The resulting clusters are consistent with the structure of Borsa Istanbul. Both MST and PMFG filtration of the clusters involve agents of Financials sector as the leading elements. To estimate the parameters of the model, we use graph theoretical concepts such as vibrational potentials and disparity measure of respected PMFGs.

By the computed parameters, we use Adomian decomposition method to obtain a solution of the model. This solution show us that for different fractional dimensions α, the model always reaches to an equilibrium state. For the lesser values of fraction rate α, agents reach to an equilibrium state relatively slower. Besides, the flows of investments tend to be in same characteristics. For the greater α values, agents reach to an equilibrium state faster and similarly the flows of investments tend to be in same characteristics. The model keeps the memory of the investment in best for $0.4 \le \alpha \le 0.6$. This results shows us that the fractional interaction of financial agents is consistent with reality when autocorrelations are discarded.

As neoclassical liberal theory of economics states, markets always seek for an equilibrium state. Hence, the model we present with fractional derivative is consistent with the real data of Borsa Istanbul Stock Exchange Market. We also believe that these kind of models can provide useful information for understanding and prediction of the global economic crisis.

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