Communications in Mathematics and Applications
Vol. 12, No. 2, pp. 295–302, 2021
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v12i2.1490

Research Article

On Rings Whose Quasi-Projective Modules Are Projective or Semisimple

Nil Orhan Ertaş *[1] and Ummahan Acar[2]

[1] *Department of Mathematics, Bursa Technical University, Bursa, Turkey*
[2] *Department of Mathematics, Muğla Sıtkı Koçman University, Muğla, Turkey*

Received: December 3, 2020 **Accepted:** February 25, 2021

Abstract. For two modules M and N, $P_M(N)$ stands for the largest submodule of N relative to which M is projective. For any module M, $P_M(N)$ defines a left exact preradical. It is given some properties of $P_M(N)$. We express $P_M(N)$ as a trace submodule. In this paper, we study rings with no quasi-projective modules other than semisimples and projectives, that is, rings whose quasi-projectives are either projective or semisimple (namely *QPS-ring*). Semi-Artinian rings and rings with no right p-middle class are characterized by using this functor: a ring R right semi-Artinian if and only if for any right R-module M, $P_M(M) \leq_e M$.

Keywords. Projective module; p-poor module; Projectivity domain; Semi-Artininan ring

Mathematics Subject Classification (2020). 16D10; 16D40; 16D50; 16D60; 16D70; 16D80

1. Introduction and Preliminaries

The purpose of this paper is to initiate the study of new left exact preradical and their various related concepts. Our rings will be associative with identity, and modules will be unitary right modules, unless stated otherwise. Let *Mod-R* denotes the category of all right R-modules. The class of all semisimple right R-modules will be denoted by *SSMod-R*. It is clear that for any R-module M, we have $SSMod\text{-}R \subseteq Mod\text{-}R$.

***Corresponding author:** orhannil@yahoo.com

Given a ring R and two R-modules M and N, M is said to be N-injective if, for any submodule A of N, every element of $Hom_R(A,M)$ extends to some element of $Hom_R(N,M)$. If M is M-injective, M is called quasi-injective. $In^{-1}(M)$ is the class of all modules X for which M is X-injective is denoted by the domain of injectivity of M (see [2]). In [1], poor modules were introduced whose injectivity domains are only semisimple modules. In [3,5], it was studied rings whose modules are injective or poor, namely rings with no right middle class. In [6], $i_M(N)$ was defined as follows: the sum of submodules A of N such that M is A-injective and by using this functor, semi-Artinian rings and the rings with no right middle class were investigated. M is said to be N-projective if, for any submodule A of N, every element of $Hom(M,N/A)$ lifts to some element of $Hom(M,N)$. If M is M-projective, M is called a quasi-projective module. $\mathfrak{P}^{-1}(M)$ denotes the domain of projectivity of M, namely, the class of all modules N for which M is N-projective (see [2]). Clearly, M is projective if $\mathfrak{P}^{-1}(M) = Mod\text{-}R$. In other words, M is projective if its projectivity domain is as large as it can be. In [9], the authors studied the class of modules M whose domain of projectivity is the smallest possible (that is $\mathfrak{P}r^{-1}(M) = SSMod\text{-}R$). They called these modules projectively poor (or p-poor modules). In [9], they proved the existence of p-poor modules for an arbitrary ring. There exists two possible domains of projectivity: semisimple modules and all modules. One may consider rings R over which all right R-modules are either projective or projectively poor. Those rings are called rings with no right p-middle class are defined in [9]. Since every module over a semisimple Artinian ring is projective, semisimple Artinian rings come up as the simplest type of those rings. Rings with no right p-middle class are not necessarily semisimple Artinian. Indeed, a quasi-Frobenius ring R with homogeneous right socle and $J(R)^2 = 0$ has no right p-middle class(see [9, Example 3.12]).

Let M, N be two modules over the ring R. In this paper we will write $P_M(N)$ for the sum of submodules A of N such that M is A-projective. In the first section we will give some properties of $P_M(N)$. We will show that $P_M(N)$ is the left exact preradical on the category $Mod\text{-}R$, and also we show that it is the largest submodule of N relative to which M is projective. The submodule $\sum\{Im\, f : f \in Hom_R(M,N)\}$ of N will be denoted $Tr_R(M,N)$ and is called the trace of M in N. We also give the relation between $Tr_R(M,N)$ and $P_M(N)$.

There exists two obvious classes of quasi-projective modules are those of semisimples and projectives. In this paper, we study rings with quasi-projective modules other than semisimples and projectives, that is, rings whose quasi-projectives are either projective or semisimple (namely, right QPS-rings). We investigate the relation between the rings with no right p-middle class and the QPS-rings. In [10], they gave the characterization of QPS-ring. We extend this result by using the functor P_M. These rings have been studied extensively in recent years [3–6, 9, 13].

For a ring R, $J(R)$, $Soc(R_R)$ will respectively denote the Jacobson radical, right socle of R. We use $\leq$, $\leq_e$, $\leq_d$ to denote the relation submodule, essential submodule, and direct summand, respectively. A module is called semi-Artinian if every homomorphic image of it has essential socle. A ring R is called right semi-Artinian if R_R is semi-Artinian. For basic terminology, concepts and results not mentioned here, we refer the reader [2, 8, 11, 12].

2. Some Properties of Preradical

In this section, we will give some properties of the left exact predical P_M for any right R-module M.

Definition 2.1. Let M and N be two right R-modules. We define $P_M(N)$ for the sum of submodules A of N such that M is A-projective.

Lemma 2.2. *Let M and N be two right R-modules. Then $P_M(N)$ is a fully invariant submodule of N.*

Proof. Let $f : N \to N$ be any homomorphism and A be a submodule of N such that M is A-projective. Now, we will show that M is $f(A)$-projective. Take the following diagram:

$$\begin{tikzcd} & & M \arrow[d, "g"] & \\ A \arrow[r, "f"'] & f(A) \arrow[r, "\pi"'] & X \arrow[r] & 0 \end{tikzcd}$$

Since M is A-projective, there exists a homomorphism $h : M \to A$ such that $\pi f h = g$. Hence g lifts to fh. Therefore $P_M(N)$ is a fully invariant submodule of N. □

Lemma 2.3. *Let M and N be two right R-modules. Then $P_M(N)$ is the largest submodule of N relative to which M is projective.*

Proof. Let $P_M(N) \le X \le N$ and M is X-projective. Clearly, $X \le P_M(N) \le X \le N$. Then, we have $P_M(N) = X$. □

Lemma 2.4. *Let M and N be two right R-modules. Then*

$$P_M(N) = \{x \in N \mid M \text{ is } xR\text{-projective}\}.$$

Proof. Say $X = \{x \in N \mid M$ is xR-projective$\}$. Let $A \le N$ and M is A-projective. Then M is aR-projective for all $a \in A$. Then $a \in X$. This implies that $A \subseteq X$. Then $P_M(N) \subseteq X$. For the converse, let $x \in X$. Since $xR \le N$ and M is xR-projective, $x \in P_M(N)$. □

Lemma 2.5. *Let M and N be two right R-modules. Then $P_M(N)$ is a left exact preradical on the category Mod-R.*

Proof. Clearly, $P_M(N)$ is a submodule of N. Let N' be a right R-module and $f : N \to N'$ be any homomorphism. Take A be a submodule of N such that M is A-projective. Then M is $f(A)$-projective. Hence $f(P_M(N)) \subseteq P_M(N')$. Let L be a submodule of N. We will show that $P_M(L) = L \cap P_M(N)$. Clearly, $P_M(L) \subseteq L \cap P_M(N)$. For converse, let $x \in P_M(N) \cap L$. Then $x \in L$ and M is xR-projective. Then $x \in P_M(L)$. □

Lemma 2.6. *Let M_1 and M_2 be two right R-modules and A be any module. Then*

$$P_A(M_1 \oplus M_2) = P_A(M_1) \oplus P_A(M_2).$$

Proof. Let $x \in P_A(M_1) \oplus P_A(M_2)$. Then $x = m_1 + m_2$, where $m_i \in M_i$ for $i = 1,2$ such that A is m_iR-projective. Then, clearly A is $m_1R \oplus m_2R$-projective. Then A is $(m_1 + m_2)R$-projective. Hence $x \in P_A(M_1 \oplus M_2)$. For the converse, let $x \in P_A(M_1 \oplus M_2)$. There exist $m_i \in M_i$ such that $x = m_1 + m_2$ and A is xR-projective. Consider the obvious projection $\pi_i : M_1 \oplus M_2 \to M_i$ for each $i = 1,2$. Since A is xR-projective, then A is $\pi_i(xR)$-projective for $i = 1,2$. Then A is m_iR-projective for each $i = 1,2$. Therefore, $x \in P_A(M_1) \oplus P_A(M_2)$. □

Corollary 2.7. *Let $\{M_i \mid i = 1,2 \dots n\}$ be the collection of right R-modules and A be any right R-modules. Then*

$$P_A(\oplus_{i=1}^n M_i) = \oplus_{i=1}^n P_A(M_i).$$

Proposition 2.8. *Let $\{M_i \mid i \in I\}$ be the collection of right R-modules and A be any right R-modules. Then*

$$P_A(\oplus_{i\in I} M_i) = \oplus_{i\in I} P_A(M_i).$$

Proof. Let $x \in P_A(\oplus_{i\in I} M_i)$. There exists a finite $J \subseteq I$ such that $x \in \oplus_{i\in J} M_i$ and A is xR-projective. By Corollary 2.7, $P_A(\oplus_{i\in J} M_i) = \oplus_{i\in J} P_A(M_i)$. Then $x \in \oplus_{i\in I} P_A(M_i)$. For the other direction, let $x \in \oplus_{i\in I} P_A(M_i)$. Again there exists a finite $J \subseteq I$ such that $x \in \oplus_{i\in J} P_A(M_i) = P_A(\oplus_{i\in J} M_i)$. Hence $x \in P_A(\oplus_{i\in I} M_i)$. □

Lemma 2.9. *Let A and B be two right R-modules. Then for any right R-module M,*

$$P_{A\oplus B}(M) = P_A(M) \cap P_B(M).$$

Proof. Let $m \in P_{A\oplus B}(M)$. Then $A \oplus B$ is mR-projective. By the properties of projectivitiy, A and B are mR-projective. Therefore, $m \in P_A(M) \cap P_B(M)$. For the converse, let $x \in P_A(M) \cap P_B(M)$. Then A and B are xR-projective. Now, we will show that $A \oplus B$ is xR-projective. Consider the following diagram:

$$\begin{tikzcd}
 & A,B \arrow[d, "i_A"', "i_B"] & \\
 & A \oplus B \arrow[d, "\gamma"] & \\
xR \arrow[r, "\pi"'] & xR/T \arrow[r] & 0
\end{tikzcd}$$

Since A and B are xR-projective, then there exist homomorphisms $f : A \to xR$, $g : B \to xR$ such that $\pi f = \gamma i_A$ and $\pi g = \gamma i_B$. Hence γ lifts to $f + g$. □

Corollary 2.10. *Let A and B be two right R-modules. Then we have*

$$P_{A\oplus B}(A \oplus B) = P_A(A \oplus B) \cap P_B(A \oplus B) = [P_A(A) \oplus P_A(B)] \cap [P_B(A) \oplus P_B(B)]$$

or

$$P_{A\oplus B}(A \oplus B) = P_{A\oplus B}(A) \oplus P_{A\oplus B}(B) = [P_A(A) \cap P_B(A)] \oplus [P_A(B) \cap P_B(B)].$$

For two modules M and N over a ring R, $Tr(M,N) = \sum_{f \in Hom(M,N)} f(M)$ is called the trace of M in N.

Lemma 2.11. *Let M and N be two modules over a ring R. Then*

$$P_M(N) = \sum_{A \in \mathfrak{P}^{-1}(M)} Tr(A,N).$$

Proof. Let $A \leq N$ and M be A-projective. Consider the inclusion map $i : A \mapsto N$. Then $A = i(A) \subseteq Tr(A,N) \subseteq \sum_{A \in \mathfrak{P}^{-1}(M)} Tr(A,N)$. Hence $P_M(N) \subseteq \sum_{A \in \mathfrak{P}^{-1}(M)} Tr(A,N)$. Now let M be A-projective. Consider $Tr(A,N)$. Let $f : A \mapsto N$ be a homomorphism. Clearly, M is $f(A)$-projective. Since $f(A) \leq N$, $f(A) \subseteq P_M(N)$. Hence we get desired result. □

Lemma 2.12. *Let M and N be modules. Then if M is N-projective, then $Tr(N,M) \subseteq P_M(N)$.*

Proof. **Clear** by Lemma 2.11. □

3. Rings Whose Quasi-Projective Modules Are Projective or Semisimple

In this section, we will give characterization of QPS-rings by using the funtor $P_M(N)$ for any modules M and N.

Definition 3.1. If every quasi-projective modules are projective or semisimple, the ring R is called QPS-ring.

Example 3.2 ([10, Proposition 4.15]). Let $R = \begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$ be a QPS-ring. Since $J(R) = \begin{pmatrix} 0 & 0 \\ \mathbb{R} & 0 \end{pmatrix}$ which does not contain two sided ideal.

Example 3.3 ([9, Example 3.12]). Let $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$, where K is a field. This ring is a QPS-ring.

Before giving the characterization of QPS-ring, we just remember following theorem.

Theorem 3.4 ([7, Theorem 3.10]). *Let $\{e_1, e_2, \dots, e_m\}$ be a basic set of primitive idempotents of a semiperfect ring R. Then for every projective R-module P_R there exist sets $A_1, \dots A_m$, uniquely determined up to cardinality, such that $P_R \cong e_1 R^{(A_1)} \oplus \cdots \oplus e_m R^{(A_m)}$.*

Theorem 3.5. *Let R be a right perfect ring. The following are equivalent:*

(i) *R has no right p-middle class,*

(ii) *For any two right R-modules M and N, $P_M(N) = Soc(N)$ or N.*

(iii) *For any right R-module M, $P_M(M) = Soc(M)$ or M.*

(iv) *R is a QPS-ring.*

(v) *Every quasi-projective module is projective or p-poor.*

Proof. (i)⇒(ii): Any ring with no right p-middle class satisfies the (ii) obviously.

(ii)⇒(iii), (i)⇒(iv) and (iv)⇒(v) are obvious.

(ii)⇒(i): Assume the ring R has right p-middle class. Then there exists a non p-poor and non projective module A. Since A is not p-poor there exists a nonsemisimple cyclic module $xR \in \mathfrak{P}^{-1}(A)$. Since R is right perfect, there exists a projective cover of A, say (P,f). By Theorem 3.4, $P \cong e_1R^{(A_1)} \oplus \cdots \oplus e_mR^{(A_m)}$, where $\{e_1,e_2,\dots,e_m\}$ be a basic set of primitive idempotents. By assumption, $P_A(P) = P$ or $P_A(P) = Soc(P)$. Firstly, assume that $P_A(P) = P$. Then $P_A(P) = P_A(e_1R^{(A_1)} \oplus \dots e_mR^{(A_m)}) = \oplus_{i=1}^n e_iR^{(A_i)}$. By the Krull-Schmidt-Remak-Azumaya Theorem, $P_A(e_iR) = e_iR$ for all $i = 1,2,\dots,m$. R can be written as a copies of direct sums of $e_1R,\dots e_mR$. Then by Corollary 2.8, $P_A(R) = R$. This implies that A is projective. This is a contradiction. Now, assume that $P_A(P) = Soc(P)$. Consider the module $P \oplus xR$. $P_A(P \oplus xR) = P_A(P) \oplus P_A(xR) = Soc(P) \oplus xR$, since $xR \in \mathfrak{P}^{-1}(A)$. By assumption, $P_A(P \oplus xR) = P \oplus xR$ or $P_A(P \oplus xR) = Soc(P) \oplus xR$. If $P_A(P \oplus xR) = P \oplus xR = Soc(P) \oplus xR$, then P is semisimple. If $P_A(P \oplus xR) = Soc(P) \oplus Soc\, xR = Soc(P) \oplus xR$, then xR is semisimple. These are contradiction.

(iii)⇒(i): Let A be a nonsemisimple quasiprojective module and (P,f) be a projective cover of A. Put $M = A \oplus P$. By assumption, $P_M(M) = M$ or $Soc(M)$. First assume that $P_M(M) = Soc(M)$. By Lemma 2.6, $P_M(A \oplus P) = P_M(A) \oplus P_M(P) = Soc(M) = Soc(A) \oplus Soc(P)$. $P_M(A) = A \cap P_M(M) = A \cap Soc(M) = Soc(A) = P_{A\oplus P}(A) = P_A(A) \cap P_P(A)$ by Lemma 2.9. Since A is quasi projective and P is projective $P_A(A) = A$ and $P_P(A) = A$. This implies that $A = Soc(A)$. It is a contradiction. Now, assume that $P_M(M) = M = A \oplus P$. This implies that M is P-projective. By properties projectivity, A is P-projective. This forces to A is projective. By [10, Proposition 4.9], R has no p-middle class.

(v)⇒(iv): Let M be any quasi-projective module but not projective. By assumption M is p-poor. Then M is semisimple. □

Proposition 3.6. *The following are equivalent for a ring R*

(i) *For any two right R-module M and $N \neq 0$, $P_M(N) \neq 0$;*

(ii) *For any two right R-module M and N, $P_M(N) \leq_e N$;*

(iii) *For any right R-module M, $P_M(M) \leq_e M$;*

(iv) *R is right semi-Artinian.*

Proof. (iv)⇒(i): Take two nonzero right R-module M and N. Since $Soc(N) \subseteq P_M(N)$ and $Soc(N) \neq 0$, we have $P_M(N) \neq 0$.

(i)⇔(ii): Take two right R-modules Let $T \cap P_M(N) = 0$ for $T \leq N$. Since P_M is left exact radical $P_M(T) = P_M(N) \cap T = 0$. If $T \neq 0$, then $P_M(T) \neq 0$ by (i). This is a contradiction. Hence $T = 0$. Hence $P_M(N) \leq_e N$.

(ii)⇒(iii): It is obvious.

(iii)⇒(ii): Let M and N be two modules with $N \neq 0$. By assumption and Lemma 2.6 and 2.9,

$$P_{M\oplus N}(M \oplus N) = [P_M(M) \oplus P_M(N)] \cap [P_N(M) \oplus P_N(N)] \leq_e M \oplus N.$$

This implies that $P_M(N) \leq_e N$.

(i)⇒(iv): Let N be a nonzero module. By [9, Propositon 2.5], there exists a p-poor module M. Then $P_M(N)$ is semisimple and by our assumption nonzero. Then R is right semi-Artinian. □

Proposition 3.7. *The following are equivalent for a ring R:*

(i) *$Soc(R_R)$ is an essential ideal of R;*

(ii) *for any right R-module M, $P_M(R) \leq_e R_R$.*

Proof. (i)⇒(ii): It is obvious.

(ii)⇒(i): By [9, Proposition 2.4], there exists a p-poor module M. Then $P_M(R)$ is semisimple and by assumption, essential in R_R, thus yielding the conclusion. □

A preradical r on *Mod-R* is called costable if $r(P)$ is a direct summand for all projective modules P.

Theorem 3.8. *Let R be a ring and $Soc(R_R) \leq_e R$. Then the following are equivalent:*

(i) *for any right R-modules M and N, $P_M(N) \leq_d N$;*

(ii) *for any right R-module M, $P_M(M) \leq_d M$;*

(iii) *for any right R-module M, P_M is costable;*

(iv) *every left exact preradical on Mod-R is stable;*

(v) *R is semisimple Artinian.*

Proof. (i)⇒(ii), (v)⇒(i), (v)⇒(iv), and (iv)⇒(iii) are obvious.

(ii)⇒(iii): Let P be a projective module and M be any module. Consider the module $P \oplus M$. By assumption $P_{M\oplus P}(M \oplus P) \leq_d M \oplus P$. Then by Lemma 2.9, $P_{M\oplus P}(M \oplus P) = [P_M(M) \cap P_P(M)] \oplus [P_M(P) \cap P_M(P)] = (P_M(M) \cap M) \oplus (P_M(P) \cap P) = P_M(M) \oplus P_M(P)$ since P is projective. This implies that $P_M(P) \leq_d P$.

(iii)⇒(v): By [9, Proposition 2.4], there exists a p-poor module M. Then $P_M(R) = Soc(R_R)$. By assumption, $P_M(R) \leq_d R$. Hence we get $Soc(R_R) = R$. □

Acknowledgment

Authors would like to thank Prof. Dr. Derya Keskin Tütüncü for pointing out the Lemma 2.11.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] A.N. Alahmadi, M. Alkan and S.R. López-Permouth, Poor modules: the opposite of injectivity, *Glasgow Mathematical Journal* **52**(A) (2010), 7 – 17, DOI: 10.1017/S001708951000025X.

[2] F.W. Anderson and K.R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York (1974).

[3] P. Aydoğdu and B. Saraç, On Artinian rings with restricted class of injectivity domains, *Journal of Algebra* **377** (2013), 49 – 65, DOI: 10.1016/j.jalgebra.2012.11.027.

[4] H.Q. Dinh, C.J. Holston and D.V. Huynh, Quasi-projective modules over prime hereditary noetherian V-rings are projective or injective, *Journal of Algebra* **360** (2012), 87 – 91, DOI: 10.1016/j.jalgebra.2012.04.002.

[5] N. Er, S.R. López-Permouth and N. Sökmez, Rings whose modules have maximal or minimal injectivity domains, *Journal of Algebra* **330**(1) (2011), 404 – 417, DOI: 10.1016/j.jalgebra.2010.10.038.

[6] N. Er, Rings characterized via a class of left exact preradicals, *Proceedings of the Edinburgh Mathematical Society* **59**(3) (2016), 641 – 653, DOI: 10.1017/S0013091515000206.

[7] A. Facchini, *Module Theory, Endomorphism rings and direct sum decompositions in some classes of modules*, Birkhäuser-Verlag (1998).

[8] K. Goodearl, *Singular Torsion and the Splitting Properties*, Memoirs of the American Mathematical Society, 124, 89 pages, (1972), https://bookstore.ams.org/memo-1-124.

[9] C. Holston, S.R. Lopez-Permouth and N.O. Ertaş, Rings whose modules have maximal or minimal projectivity domain, *Journal of Pure and Applied Algebra* **216** (2012), 673 – 678, DOI: 10.1016/j.jpaa.2011.08.002.

[10] S.R. López-Permouth and J.E. Simental, Characterizing rings in terms of the extent of the injectivity and projectivity of their modules, *Journal of Algebra* **362** (2012), 56 – 69, DOI: 10.1016/j.jalgebra.2012.04.005.

[11] S.H. Mohamed and B.J. Müller, *Continuous and discrete modules*, London Mathematical Society Lecture Note Series, 147, Cambridge University Press, Cambridge (1990).

[12] K.M. Rangaswamy and N. Vanaja, Quasi projectives in abelian and module categories, *Pacific Journal of Mathematics* **43**(1) (1972), 221 – 238, https://projecteuclid.org/download/pdf1/euclid.pjm/1102959656.

[13] B. Saraç, On rings whose quasi-injective modules are injective or semisimple, https://archive.org/details/qis-rings.