Scientia Iranica E (2021) 28(6), 3719–3742

SCIENTIA IRANICA

Sharif University of Technology
Scientia Iranica
Transactions E: Industrial Engineering
http://scientiairanica.sharif.edu

Three-valued soft set and its multi-criteria group decision making via TOPSIS and ELECTRE

E. Akçetin[a] and H. Kamacı[b,*]

a. *Department of Accounting and Financial Management, Seydikemer School of Applied Sciences, Muğla Sıtkı Koçman University, Muğla, Turkey.*
b. *Department of Mathematics, Faculty of Science and Arts, Yozgat Bozok University, Yozgat, Turkey.*

Received 24 October 2019; received in revised form 3 January 2020; accepted 20 April 2020

KEYWORDS
Soft set;
Three-valued soft set;
Choice value;
Decision making;
TOPSIS;
ELECTRE.

Abstract. The main objective of the present study is to point to the generalization of Molodtsov's approach to soft sets obtained by passing from the classical two-valued logic underlying those sets to a three-valued logic where the third truth value can usually be interpreted as either non-determined (i.e., between true and false) or unknown. This extension of soft set approach allows a more intuitive and clearer representation of various decision-making problems involving incomplete or uncertain information. In other words, it is a viable approach to analyze soft-set-based multi-criteria-group decision-making problems in the absence of adequate information resulting from the inability to determine the data. In this regard, this study introduced the concept of three-valued soft set and some of its operations and products. In addition, the formulas required to calculate all possible choice values were proposed for each object in the (weighted) three-valued soft sets and their respective decision values were calculated. Both Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) and ELimination Et Choice Translating REality (ELECTRE) methods were modified to deal with multi-criteria group decision problems and then, three-valued soft-set-based decision-making algorithms were constructed. To demonstrate the practicality of these algorithms, the examples adopted from the decision problems in real life were addressed. Lastly, some aspects of the efficiency of the proposed algorithms were discussed using computational experiments.

1. Introduction

Classically, a logic is two-valued (Boolean) if every proposition is either false ("0") or true ("1"). In 1930, Łukasiewicz [1] initiated a three-valued logic, i.e., a natural extension of the two-valued logic with three truth (logical) values indicating false ("0"), true ("1"), and some indeterminate third values ("$\frac{1}{2}$" as something in the middle between true and false). Further, he pioneered the conceptual form and basic ideas of three-valued logic. By interpreting the intuition of Łukasiewicz three-valued logic from different perspectives, two-valued logic was extended to three-valued logic in different ways [2–5]. Based on their choice of basic connectives, they were different from a syntactic and proof-theoretic point of view. Although the idea of fuzzification of logic was envisaged by several researchers in the years following 1920, the concept of fuzzy logic in which the truth values of

*. *Corresponding author. Tel.: +90 354 242 10 21; Fax: +90 354 242 10 22*
E-mail addresses: eyup.akcetin@mu.edu.tr (E. Akçetin); huseyin.kamaci@hotmail.com, huseyin.kamaci@bozok.edu.tr (H. Kamacı)

doi: 10.24200/sci.2020.54715.3881

variables might be any real number between 0 and 1, both inclusive, was explicitly and crisply proposed by Zadeh [6] in 1965. Fuzzy logic is based on the investigation and observation that people take into account while making decisions based on uncertain and non-numerical information. Fuzzy set, which is a generalization of the crisp set based on the two-valued logic, is the mathematical means of representing vagueness and imprecise information. This set is described by a membership function that assigns to each object a degree of membership ranging between 0 and 1. Pawlak [7] defined a rough set that can be considered as a new area of uncertainty mathematics closely related to set theory. This set is a formal approximation of a crisp set in a pair of sets which give the lower and upper approximations of the original set. The approximation spaces of rough set theory are sets with multiple memberships, while fuzzy sets are concerned with partial memberships. These sets are combined to derive different variations such as the fuzzy rough set and rough fuzzy set. In 1999, Molodtsov [8] described the soft set based on two-valued logic as a mathematical tool dealing with parametric data which were imprecise or uncertain in nature. In 2003, Maji et al. [9] published a study on the operations of soft sets. Later, the operational laws of the soft sets were derived [10–14]. In addition, many authors have described and discussed different types of soft sets such as bijective soft set [15], exclusive disjunctive soft set [16], bipolar soft set [17], inverse soft set [18,19], etc.

Decision making, which is one of the issues that triggers uncertainty, is a frequently encountered problem in many commercial and scientific fields, even in every stage of daily life. For deterministic modeling of decision-making problems, many mathematical techniques such as Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), AHP, VIKOR, ELimination Et Choice Translating (ELECTRE), and PROMETHEE have been developed. These mathematical techniques were adopted for decision-making based on the fuzzy set and some of its extensions [20–26]. In addition to the fuzzy modelling of TOPSIS, ELECETRE, etc., many algorithmic solutions were proposed for decision-making in a fuzzy environment [27–31]. In 2002, Maji et al. [32] reported that soft sets could be used to solve decision-making problems involving parametric data based on two-valued logic. In the following years, many soft decision algorithms were created and their applications to problems in real life were specified [33–37]. Moreover, Eraslan [38] proposed a decision-making procedure through the classical method of TOPSIS on the soft sets. In this regard, he pointed out that classical decision-making techniques could be successfully applied to the soft-set-based decision-making.

In 2008, Avron and Konikowska [39] explored the idea of describing Pawlak's rough set using three-valued logic. This paradigm presents a different perspective in the interpretation of issues involving indeterminate or unknown data in many fields. As with the rough set, three-valued logic emerges in many real-world scenes that are included in the scope of the soft set, and it is difficult to deal with such broadly scoped issues whose third truth value is "undetermined". In this study, a completely designed approach to soft set was discussed using three-valued logic. Thus, in practice, we aim to overcome the difficulties that include a third truth value caused by uncertain or unknown origin, in addition to the truth values of a two-valued logic. The present study was conducted based on the idea of proposing the notion of three-valued soft set to describe a soft set using three-valued logic. The fusion of three-valued logic into the soft set suggests a clearer and intuitive way to explain different issues under incomplete or uncertain information. In this regard, the main focus was put on decision-making based on this type of soft set and offering different algorithmic solutions.

The rest of this paper is organized as follows. Section 2 gives an outline of soft set theory. Section 3 elaborates the motivation to reinterpret soft sets using three-valued logic. Section 4 calculates the choice value of an object in (weighted) three-valued soft sets and accordingly, proposes two algorithms. Section 5 creates a multi-criteria group decision-making algorithm based on the modified TOPSIS on three-valued soft sets. Section 6 proposes a three-valued soft decision-making algorithm via the modified ELECTRE, which is based on three fundamental objectives called choosing, sorting, and ranking. In addition, some examples were given to analyze the performance of the algorithms emerging in these two sections. Section 7 solves the matching numerical examples to compare the results of the proposed algorithms, thus showing that they are convincing. The last section presents the concluding remarks and suggests plans for further research.

2. Preliminaries

As a preparatory opening for new concepts, this section elaborates some relevant arguments of soft set, two-valued logic, and three-valued logic.

Consider the soft set theory first. In 1999, Molodtsov [8] introduced soft set theory as a useful way of classifying objects based on parametric data. In 2010, Çağman and Enginoğlu [40] recreated soft sets to make their operations more practical in some cases. Maji et al. [32] put forward that the soft set could be represented in a tabular form. They also demonstrated that soft sets were the parametric sets created based on two-valued logic (i.e., 0 as false, 1 as true). Now, recall the definition of soft set.

From now on, $\mathfrak{X}$ is a universal set, $\mathcal{P}$ is a parameter set, and $Q \subseteq \mathcal{P}$.

Definition 2.1 [8,40]. Assume that $\mathfrak{X}^{\langle\{0,1\}\rangle}$ denotes the set of all functions from $\mathfrak{X}$ to $\{0,1\}$. A pair $(f_Q, \mathcal{P}) = F_Q$ is called a soft set over $\mathfrak{X}$ when the mapping f_Q is defined by $f_Q : \mathcal{P} \to \mathfrak{X}^{\langle\{0,1\}\rangle}$, where for all $p_j \in \mathcal{P}$, the approximate function $f_Q(p_j)$ is shown in Box I.

Example 2.1. In daily life, many operations such as shopping, booking, money transfer, etc., can be done through online websites. Recently, some websites have made it feasible to reserve hotel rooms for trips and holidays; take Booking.com and Expedia.com as examples. Now, consider the following problem that one may encounter when making a hotel room reservation.

Let $\mathfrak{X} = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of five hotels that are available for booking a room. Let $\mathcal{P} = \{p_1 = price, p_2 = location, p_3 = amenities, p_4 = satisfaction\}$ be a set of the choice parameters. Then, we can create the following soft set over $\mathfrak{X}$:

$$\begin{aligned} T_{\mathcal{P}} = \{&(p_1, \{x_1^{\langle 0\rangle}, x_2^{\langle 1\rangle}, x_3^{\langle 1\rangle}, x_4^{\langle 0\rangle}, x_5^{\langle 0\rangle}\}),\\ &(p_2, \{x_1^{\langle 1\rangle}, x_2^{\langle 1\rangle}, x_3^{\langle 0\rangle}, x_4^{\langle 0\rangle}, x_5^{\langle 1\rangle}\}),\\ &(p_3, \{x_1^{\langle 1\rangle}, x_2^{\langle 1\rangle}, x_3^{\langle 0\rangle}, x_4^{\langle 1\rangle}, x_5^{\langle 0\rangle}\}),\\ &(p_4, \{x_1^{\langle 0\rangle}, x_2^{\langle 1\rangle}, x_3^{\langle 1\rangle}, x_4^{\langle 0\rangle}, x_5^{\langle 1\rangle}\})\}. \end{aligned}$$

For the first pair in this soft set, it can be interpreted that the prices of hotels x_2 and x_3 are suitable for us, while the prices of hotels x_1, x_4, and x_5 are not suitable. Other pairs can be interpreted similarly.

In 1930, Łukasiewicz [1] put forward (Lukasiewicz) three-valued logic by extending two-valued logic called Boolean logic. Immediately after the above author, many authors have exhibited their interest in the idea of three-valued logic and its operations [2,4,39,41,42]. They argued that since two-valued logic (Boolean logic) could cover all kinds of scientific investigations, three-valued logic might be useful as a basis for a number of useful reasoning tasks. Boolean connectives can be extended to three-valued logic in different ways. In other words, the third truth value can be explained in different ways that are different from true and false. Ciucci and Dubois [43] listed these ways as follows:

- *Possible*: this explanation was proposed by Łukasiewicz [1] and Borowski [2] the pioneer of three-valued logic. A proposition is regarded as “Possible” if its truth value will be known only in the future;
- *Unknown*: This explanation was proposed by Kleene [4] in 1952. A proposition is “Unknown” if its truth value cannot be computed for some reasons (for instance, it is too time-consuming to compute);
- *Inconsistent*: The third value stands for a proposition that is both true and false, and also it is the dual of “Unknown” in some sense;
- *Half-true*: This is the typical of fuzzy logic [3]. The intuition is that for some propositions, truth is a importance degree. For instance, Shadowed set in [44,45] is based on the idea of turning fuzzy set into three-valued one;
- *Undefined*: This is another explanation of Kleene [4]. The undefined state corresponds to the selection of the argument of the function outside its definition domain. A proposition is “Undefined” if its truth value involves undefined atoms;
- *Irrelevant*: The idea behind it is that propositions are not applicable in some possible worlds.

In 1960, Skolem [5] initiated a set theory based on a certain three-valued logic. In this set theory, the variables such as $p, q, r, ...$ take three values: $0, \frac{1}{2}, 1$. We may interpret 0 as “false”, 1 as “true”, and $\frac{1}{2}$ as something in the middle between true and false, say “undetermined”. Moreover, Skolem presented a set of truth tables showing tree-valued logic operations like negation, disjunction and conjunction (for a detailed review, see [5]). In the literature, there are many truth tables illustrating tree-valued logic operations. However, this study focused on the truth tables proposed by Skolem [5].

Three-valued logic emerges in several real-world scenes. Three samples in the atmosphere of “undetermined” are presented in Figures 1–3. In these figures, “?” symbolizes “undetermined”, i.e., $\frac{1}{2}$.

Since the truth value of “undetermined” in Figures 1 and 2 is precisely known in the future, it can

$$f_Q(p_j) = \begin{cases} \left\{ x_i^{\langle \lambda^i_{f_{Q(p_j)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{f_{Q(p_j)}} \in \{0,1\} \right\}, & \text{if } p_j \in Q \\ \left\{ x_i^{\langle 0 \rangle} : \forall x_i \in \mathfrak{X} \right\}, & \text{if } p_j \in \mathcal{P} - Q \end{cases}$$

Box I

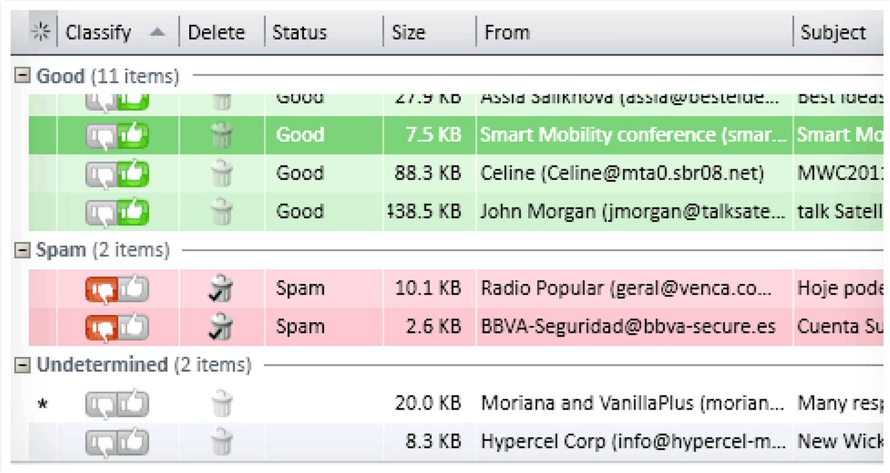

Figure 1. Three-valued logic for classifying mail in a mailbox.

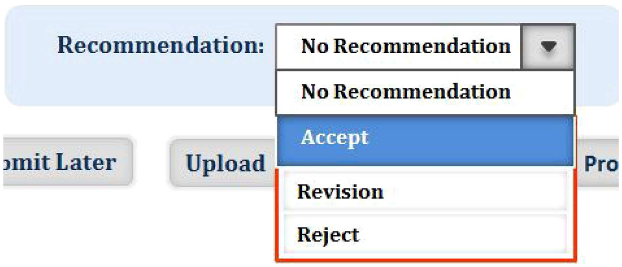

Figure 2. Three-valued logic for reviewers' recommendation in journal.

be considered as "Possible". Figure 3 can also be considered as an example for "Unknown".

3. Three-valued soft sets

This section discusses three-valued soft set which is a generalization of soft set obtained by passing from two-valued logic to three-valued one.

The soft set is a set approach proposed by two-valued logic (true and false). In daily life, while evaluating alternatives according to parameters in the decision-making process, "undetermined" mode (neither true nor false, or both true and false) sometimes arises. In such decision-making processes, the soft sets are insufficient. To overcome this shortcoming, the notion of three-valued soft set, which is a soft set based on three-valued logic (i.e., 0 as false, 1 as true, and $\frac{1}{2}$ as "undetermined") is formed.

3.1. Three-valued soft set

Definition 3.1. Assume that $\mathfrak{X}^{\langle\{0,\frac{1}{2},1\}\rangle}$ denotes the set of all functions from $\mathfrak{X}$ to $\{0, \frac{1}{2}, 1\}$. A pair $(t_Q, \mathcal{P}) = T_Q$ is called a three-valued soft set over $\mathfrak{X}$ when the mapping t_Q is defined by $t_Q : \mathcal{P} \to \mathfrak{X}^{\langle\{0,\frac{1}{2},1\}\rangle}$, where for all $p_j \in \mathcal{P}$ the equation shown in Box II is obtained.

Notation: For the parameter set $\mathcal{P}$, the set of all three-valued soft sets over $\mathfrak{X}$ is denoted by $\mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$.

Example 3.1. Assume that $\mathfrak{X} = \{x_1, x_2, x_3, x_4, x_5\}$ is a set of drugs that can be taken by pregnant

Gene Allele Symbol	Rhinosinusitis	Otitis Media	Situs Inversus	Infertifity
Ulk4^{tmlLex}	Yes	Yes	Undetermined	Undetermined
Nme5^{tmlLex}	Yes	No	No	Yes
Nme7$^{Gt(OST3I6)Lex}$	Yes	No	Yes	No
Kif27$^{Gt(OST4I9I5)Lex}$	Yes	Yes	No	Undetermined
Stk36^{tmlLex}	No	Yes	No	Yes
Dpcd$^{Gt(OST280355Lex}$, *Poll*$^{Gt(OST280355)Lex}$	Yes	No	Yes	Yes
Ak7$^{Gt(OST434404)Lex}$	Yes	No	No	Yes
Ak8$^{Gt(OSTI6378)Lex}$	No	No	No	No
4930444A02Rik$^{Gt(OST243203)Lex}$	No	No	Undetermined	Yes
Celsr2^{tmlLex}	No	No	No	No
Mboat7^{tmlLex}	No	No	No	No
Tg(FZD3)ILex	No	No	No	No

Figure 3. Three-valued logic for cilia-related lesions in hydrocephalic mice (see [46]).

$$t_Q(p_j) = \begin{cases} \left\{ x_i^{\langle \lambda^i_{t_{Q(p_j)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{Q(p_j)}} \in \{0, \frac{1}{2}, 1\} \right\}, & \text{if } p_j \in Q \\ \left\{ x_i^{\langle 0 \rangle} : \forall x_i \in \mathfrak{X} \right\}, & \text{if } p_j \in \mathcal{P} - Q \end{cases}$$

Box II

Table 1. The side effects specified in the prospectus of drugs.

	$p_1 = allergy$	$p_2 = fetal\ damage$	$p_3 = pharmacological\ effect$
x_1	✓	×	–
x_2	×	–	✓
x_3	✓	×	×
x_4	×	–	–
x_5	×	✓	×

Note: The symbols ✓, ×, and – represent Yes, No and Undetermined, respectively.

women and those of childbearing age. The drugs that can be taken by these women, can be observed without increase in the frequency of malformation or other direct or indirect harmful effects on the human fetus. Therefore, these women should carefully check their side effects while using the drug. Suppose that $\mathcal{P} = \{p_1 = allergy, p_2 = fetal\ damage, p_3 = pharmacological\ effect\}$ denotes some side effects of the drugs. Here, the following table (Table 1) can be obtained by examining the prospectus of drugs.

Based on Table 1, the following three-valued soft set is constructed:

$$T_{\mathcal{P}} = \{(p_1, \{x_1^{\langle 1 \rangle}, x_2^{\langle 0 \rangle}, x_3^{\langle 1 \rangle}, x_4^{\langle 0 \rangle}, x_5^{\langle 0 \rangle}\}),$$
$$(p_2, \{x_1^{\langle 0 \rangle}, x_2^{\langle \frac{1}{2} \rangle}, x_3^{\langle 0 \rangle}, x_4^{\langle \frac{1}{2} \rangle}, x_5^{\langle 1 \rangle}\}),$$
$$(p_3, \{x_1^{\langle \frac{1}{2} \rangle}, x_2^{\langle 1 \rangle}, x_3^{\langle 0 \rangle}, x_4^{\langle \frac{1}{2} \rangle}, x_5^{\langle 0 \rangle}\}).$$

Example 3.2. Let $\mathfrak{X} = \{x_1 = Vivo\ V17\ Pro, x_2 = OnePlus\ 7, x_3 = Xiaomi\ Redmi\ K20\ Pro, x_4 = Apple\ IPhone\ 11\}$ be a set of four mobiles, and $\mathcal{P} = \{p_1 = FM\ Radio, p_2 = Stereo\ Speakers, p_3 = Loudspeaker\}$ a set of multimedia features (attributes) that may be available on phones. Through the website "*www.91mobiles.com*" (date: 17.10.2019), we have Figure 4. Based on this figure, the following three-valued soft set can be created:

$$T_{\mathcal{P}} = \{(p_1, \{x_1^{\langle 1 \rangle}, x_2^{\langle 0 \rangle}, x_3^{\langle \frac{1}{2} \rangle}, x_4^{\langle 0 \rangle}\}),$$
$$(p_2, \{x_1^{\langle \frac{1}{2} \rangle}, x_2^{\langle 1 \rangle}, x_3^{\langle \frac{1}{2} \rangle}, x_4^{\langle 1 \rangle}\}),$$
$$(p_3, \{x_1^{\langle 1 \rangle}, x_2^{\langle 1 \rangle}, x_3^{\langle 1 \rangle}, x_4^{\langle 1 \rangle}\})\}.$$

Based on the information presented on this website, we can interpret that for the multimedia feature p_1 (FM Radio):

- The mobile x_1 (Vivo V17 Pro) has an FM Radio;
- The mobiles x_2 and x_4 (OnePlus 7 and Apple IPhone 11) do not have an FM Radio; and
- It remains "*undetermined*" whether the mobile x_3 (Xiaomi Redmi K20 Pro) has an FM Radio.

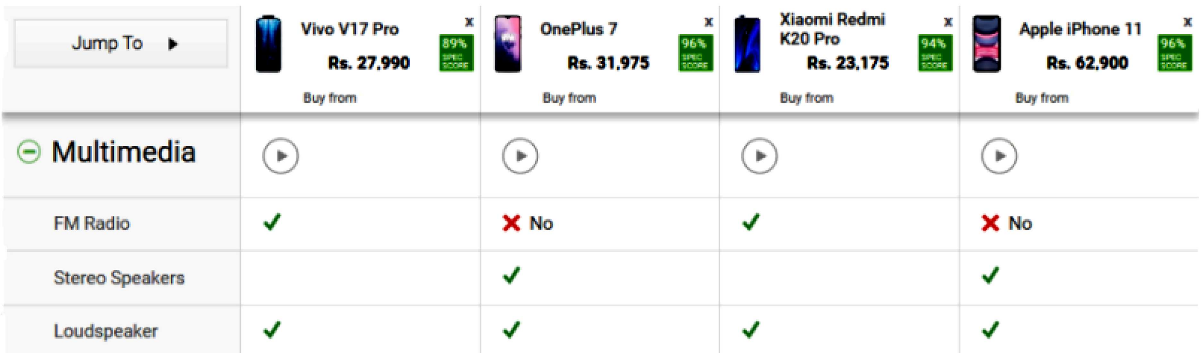

Figure 4. Comparison of some multi-media features of four mobiles.

Comment: The truth value $\frac{1}{2}$ in the three-valued logic differs from the membership degree 0.5 in the fuzzy logic. The truth value $\frac{1}{2}$ in the three-valued logic represents "undetermined", not membership. Why is this truth value between 0 and 1, not 0? 0 indicates that the given object does not certainly have the desired property, while $\frac{1}{2}$ means that the object might possibly have this property. To be specific, assume that one wants to buy any item from the mobiles x_2 and x_3 (presented in Figure 4). As a criterion for purchasing, the requirement to have only FM Radio is determined. In such situations, it is more convenient to select the mobile x_3. Since it is not known (undetermined) whether or not the mobile x_3 has an FM Radio and the mobile x_2 has no FM Radio, there is also the possibility that the mobile x_3 has an FM Radio. Indeed, we reconsider through the website "www.smartprix.com" that the mobile x_3 (Xiaomi Redmi K20 Pro) has an FM Radio.

Example 3.3. Let $\mathfrak{X} = \{x_1, x_2, x_3, x_4\}$ be a set of four investments at the disposal of the investor to invest some money and $\mathcal{P} = \{p_1 = \textit{riskless}, p_2 = \textit{security}, p_3 = \textit{tax free}, p_4 = \textit{short period}\}$ a set of parameters. For parameter subset $Q_1 = \{p_1, p_2, p_3\}$, an investor can create the following three-valued soft set over $\mathfrak{X}$:

$$T_{Q_1} = \{(p_1, \{x_1^{\langle 1\rangle}, x_2^{\langle 0\rangle}, x_3^{\langle 1\rangle}, x_4^{\langle \frac{1}{2}\rangle}\}),$$
$$(p_2, \{x_1^{\langle 0\rangle}, x_2^{\langle 1\rangle}, x_3^{\langle \frac{1}{2}\rangle}, x_4^{\langle 1\rangle}\}),$$
$$(p_3, \{x_1^{\langle 1\rangle}, x_2^{\langle \frac{1}{2}\rangle}, x_3^{\langle \frac{1}{2}\rangle}, x_4^{\langle 1\rangle}\})\}.$$

The element $(p_1, \{x_1^{\langle 1\rangle}, x_2^{\langle 0\rangle}, x_3^{\langle 1\rangle}, x_4^{\langle \frac{1}{2}\rangle}\})$ in T_{Q_1} means that:

- The investments x_1 and x_3 are risk-free;
- The investment x_2 is risky; and
- The investment x_4 is "*undetermined*" in terms of risk.

As shown in the above example, if $p_j \in \mathcal{P}\backslash Q_1$, the pair $(p_j, t_{Q_1}(p_j))$ does not need to be displayed in the structure of the three-valued soft set T_{Q_1}. However, it is known that $(p_j, \{x_i^{\langle 0\rangle} : \forall x_i \in \mathfrak{X}\})$.

Each three-valued soft set can be represented in the form of a binary table. This representation makes three-valued soft sets useful in different computer program languages as well as the practicality of calculations.

The binary tabular form of three-valued soft set T_{Q_1} in Example 3.3 is presented in Table 2.

In Table 2, each component x_{ij} represents the truth value $\lambda^i_{t_{Q(p_j)}}$ of the alternative x_i with respect to the parameter p_j, that is, in Table 2 and onward:

Table 2. The tabular form of three valued soft set T_{Q_1} in Example 3.3.

$\mathfrak{X}/P$	p_1	p_2	p_3	p_4
x_1	1	0	1	0
x_2	0	1	$\frac{1}{2}$	0
x_3	1	$\frac{1}{2}$	$\frac{1}{2}$	0
x_4	$\frac{1}{2}$	1	1	0

- $x_{ij} = 1$ means that x_i belongs to the subset of $\mathfrak{X}$ approximated by the parameter p_j;
- $x_{ij} = 0$ means that x_i does not belong to the subset of $\mathfrak{X}$ approximated by the parameter p_j; and
- $x_{ij} = \frac{1}{2}$ means that it is undetermined whether x_i belongs to the subset of $\mathfrak{X}$ approximated by the parameter p_j.

From now on, in the examples, the three-valued soft sets will be represented by the binary tables.

Definition 3.2. Let $T_Q \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. It is called:

a) An empty three-valued soft set when $t_Q(p_j) = \{x_i^{\langle 0\rangle} : \forall x_i \in \mathfrak{X}\}$ for all $p_j \in \mathcal{P}$ and it is denoted by T_Θ;

b) A Q-mid three-valued soft set when $t_Q(p_j) = \{x_i^{\langle \frac{1}{2}\rangle} : \forall x_i \in \mathfrak{X}\}$ for all $p_j \in Q$, and it is denoted by $T_{\widetilde{Q}}$. If $Q = \mathcal{P}$, the Q-mid three-valued soft set is called a mid three-valued soft set and it is denoted by $T_{\widetilde{\mathcal{P}}}$;

c) A Q-universal three-valued soft set when $t_Q(p_j) = \{x_i^{\langle 1\rangle} : \forall x_i \in \mathfrak{X}\}$ for all $p_j \in Q$, and it is denoted by $T_{\widehat{Q}}$. If $Q = \mathcal{P}$, the Q-universal three-valued soft set is called a universal three-valued soft set, and it is denoted by $T_{\widehat{\mathcal{P}}}$.

Definition 3.3. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$, then, we have:

a) T_{Q_1} is a three-valued soft subset of T_{Q_2} when $t_{Q_1}(p_j) \subseteq t_{Q_2}(p_j)$ for all $p_j \in P$ and it is denoted by $T_{Q_1} \sqsubseteq T_{Q_2}$. Here, $t_{Q_1}(p_j) \subseteq t_{Q_2}(p_j)$ for $p_j \in \mathcal{P}$ means $\lambda^i_{t_{Q1(p_j)}} \leq \lambda^i_{t_{Q2(p_j)}}$ for each $x_i \in \mathfrak{X}$.

b) T_{Q_1} and T_{Q_2} are both equal three-valued soft sets when $t_{Q_1}(p_j) = t_{Q_2}(p_j)$ for all $p_j \in P$, denoted by $T_{Q_1} = T_{Q_2}$. Here, $t_{Q_1}(p_j) = t_{Q_2}(p_j)$ for $p_j \in \mathcal{P}$ means that $\lambda^i_{t_{Q1(p_j)}} = \lambda^i_{t_{Q2(p_j)}}$ for each $x_i \in \mathfrak{X}$.

Example 3.4. Let us consider the three-valued soft set T_{Q_1} given in Table 2 of Example 3.3. In addition, the three-valued soft set T_{Q_2} is shown in Table 3. Then, $T_{Q_2} \sqsubseteq T_{Q_1}$.

Proposition 3.1. Let $T_{Q_1}, T_{Q_2}, T_{Q_3} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$.

Table 3. The tabular form of T_{Q_2} for $Q_2 = \mathcal{P}$.

$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4
x_1	$\frac{1}{2}$	0	$\frac{1}{2}$	0
x_2	0	1	$\frac{1}{2}$	0
x_3	$\frac{1}{2}$	0	0	0
x_4	0	1	1	0

i. $T_\Theta \sqsubseteq T_{\widetilde{\mathcal{P}}} \sqsubseteq T_{\widehat{\mathcal{P}}}$;

ii. $T_\Theta \sqsubseteq T_{Q_k}$ for each k;

iii. $T_{Q_k} \sqsubseteq T_{\widehat{\mathcal{P}}}$;

iv. $T_{Q_k} \sqsubseteq T_{Q_k}$ for each k;

v. $T_{Q_1} \sqsubseteq T_{Q_2}$ and $T_{Q_2} \sqsubseteq T_{Q_3} \Rightarrow T_{Q_1} \sqsubseteq T_{Q_3}$.

Proof. The proofs are obvious according to Definitions 3.2 and 3.3, hence they are omitted.□

3.2. Operations and products on three-valued soft sets

Definition 3.4. Let $T_Q \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then the complement of T_Q, denoted by $\overline{T_Q}$, is defined by the mapping $\overline{t_Q} : \mathcal{P} \to \mathfrak{X}^{(\{0,\frac{1}{2},1\})}$ such that:

$$\overline{t_Q}(p_j) = \left\{ x_i^{\langle \lambda^i_{\overline{t_Q(p_j)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{Q(p_j)}} \in \{0, \frac{1}{2}, 1\} \right\},$$

for all $p_j \in \mathcal{P}$ where:

$$\lambda^i_{\overline{t_{Q(p_j)}}} = 1 - \lambda^i_{t_{Q(p_j)}}. \tag{1}$$

Note: This definition clarifies why the truth value for "undetermined" is $\frac{1}{2}$. The negation of "undetermined" must also have the same truth value because there is no gauge of "undetermined". Accordingly, we argue that "undetermined" implies something in the middle between true and false, whose truth vale is $\frac{1}{2}$.

Table 4. The tabular form of T_{Q_2} for $Q_2 = \mathcal{P}$.

$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4
x_1	$\frac{1}{2}$	1	1	0
x_2	0	0	0	0
x_3	$\frac{1}{2}$	$\frac{1}{2}$	1	0
x_4	1	0	0	1

Table 5. The tabular form of $\overline{T_{Q_2}}$.

$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4
x_1	$\frac{1}{2}$	0	0	1
x_2	1	1	1	1
x_3	$\frac{1}{2}$	$\frac{1}{2}$	0	1
x_4	0	1	1	0

Example 3.5. We consider the universal set $\mathfrak{X}$ and parameter set $\mathcal{P}$ in Example 3.3. Furthermore, we generated the three-valued soft set T_{Q_2} given in Table 4.

The complement of T_{Q_2} is obtained and shown in Table 5.

Proposition 3.2. Let $T_Q \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$.

i. $\overline{(\overline{T_Q})} = T_Q$;

ii. $\overline{T_\Theta} = T_{\widehat{\mathcal{P}}}$;

iii. $\overline{T_{\widetilde{\mathcal{P}}}} = T_{\widetilde{\mathcal{P}}}$.

Proof. (i) We consider three-valued soft set $T_Q = (t_Q, \mathcal{P})$ over $\mathfrak{X}$. Then, we have the mapping $t_Q : \mathcal{P} \to \mathfrak{X}^{(\{0,\frac{1}{2},1\})}$ so that for all $p_j \in \mathcal{P}$ we obtain the equation shown in Box III. Based on Definition 3.4, we can write for all $p_j \in \mathcal{P}$ the equation shown in Box IV. When proceeding in a similar manner, for all $p_j \in \mathcal{P}$ we

$$t_Q(p_j) = \begin{cases} \left\{ x_i^{\langle \lambda^i_{t_{Q(p_j)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{Q(p_j)}} \in \{0, \frac{1}{2}, 1\} \right\}, & \text{if } p_j \in Q \\ \left\{ x_i^{\langle 0 \rangle} : \forall x_i \in \mathfrak{X} \right\}, & \text{if } p_j \in \mathcal{P} - Q \end{cases}$$

Box III

$$\overline{t_Q}(p_j) = \begin{cases} \left\{ x_i^{\langle 1-\lambda^i_{t_{Q(p_j)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{Q(p_j)}} \in \{0, \frac{1}{2}, 1\} \right\}, & \text{if } p_j \in Q \\ \left\{ x_i^{\langle 1-0 \rangle} : \forall x_i \in \mathfrak{X} \right\}, & \text{if } p_j \in \mathcal{P} - Q \end{cases}$$

Box IV

$$\overline{\overline{t_Q}}(p_j) = \begin{cases} \left\{ x_i^{\langle 1-(1-\lambda^i_{t_{Q(p_j)}}) \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{Q(p_j)}} \in \{0, \frac{1}{2}, 1\} \right\}, & \text{if } p_j \in Q \\ \left\{ x_i^{\langle 1-(1-0) \rangle} : \forall x_i \in \mathfrak{X} \right\}, & \text{if } p_j \in \mathcal{P} - Q \end{cases}$$

$$= \begin{cases} \left\{ x_i^{\langle \lambda^i_{t_{Q(p_j)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{Q(p_j)}} \in \{0, \frac{1}{2}, 1\} \right\}, & \text{if } p_j \in Q \\ \left\{ x_i^{\langle 0 \rangle} : \forall x_i \in \mathfrak{X} \right\}, & \text{if } p_j \in \mathcal{P} - Q \end{cases}$$

Box V

obtained the equation shown in Box V. Therefore, we have $\overline{\overline{t_Q}}(p_j) = t_Q(p_j)$ for all $p_j \in \mathcal{P}$. So, $\overline{(\overline{T_Q})} = T_Q$.

Proofs (ii) and (iii) are obvious, hence they are omitted.□

Definition 3.5. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then, the intersection of T_{Q_1} and T_{Q_2}, denoted by $T_{Q_1} \overline{\sqcap}\, T_{Q_2}$, is defined by the mapping $t_{Q_1 \overline{\sqcap} Q_2} : \mathcal{P} \to \mathfrak{X}^{(\{0,\frac{1}{2},1\})}$ such that:

$$t_{Q_1 \overline{\sqcap} Q_2}(p_j) = \left\{ x_i^{\langle \lambda^i_{t_{(Q_1 \overline{\sqcap} Q_2)(p_j)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{(Q_1 \overline{\sqcap} Q_2)(p_j)}} \in \{0, \frac{1}{2}, 1\} \right\},$$

for all $p_j \in \mathcal{P}$, where:

$$\lambda^i_{t_{(Q_1 \overline{\sqcap} Q_2)(p_j)}} = \min \left\{ \lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}} \right\}. \tag{2}$$

Proposition 3.3. Let $T_Q \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$.

i. $T_Q \,\overline{\sqcap}\, T_\Theta = T_\Theta$;
ii. $T_Q \,\overline{\sqcap}\, T_{\widehat{Q}} = T_Q$;
iii. $T_Q \,\overline{\sqcap}\, T_{\widetilde{Q}} \sqsubseteq T_Q$.

Proof. It is clear from Definitions 3.2, 3.3, and 3.5.

Definition 3.6. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then, the union of T_{Q_1} and T_{Q_2}, denoted by $T_{Q_1} \sqcup T_{Q_2}$, is defined by the mapping $t_{Q_1 \sqcup Q_2} : \mathcal{P} \to \mathfrak{X}^{(\{0,\frac{1}{2},1\})}$ such that:

$$t_{Q_1 \sqcup Q_2}(p_j) = \left\{ x_i^{\langle \lambda^i_{t_{(Q_1 \sqcup Q_2)(p_j)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{(Q_1 \sqcup Q_2)(p_j)}} \in \{0, \frac{1}{2}, 1\} \right\},$$

for all $p_j \in \mathcal{P}$, where:

$$\lambda^i_{t_{(Q_1 \sqcup Q_2)(p_j)}} = \max \left\{ \lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}} \right\}. \tag{3}$$

Proposition 3.4. Let $T_Q \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$.

i. $T_Q \sqcup T_\Theta = T_Q$;
ii. $T_Q \sqcup T_{\widehat{Q}} = T_{\widehat{Q}}$;
iii. $T_Q \sqcup T_{\widetilde{Q}} \sqsupseteq T_Q$.

Proof. It is clear from Definitions 3.2, 3.3, and 3.6.□

Proposition 3.5. Let $T_{Q_1}, T_{Q_2}, T_{Q_3} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. For all $*, \circ \in \{\overline{\sqcap}, \sqcup\}$:

i. $T_{Q_k} * T_{Q_k} = T_{Q_k}$;
ii. $T_{Q_1} * (T_{Q_2} * T_{Q_3}) = (T_{Q_1} * T_{Q_2}) * T_{Q_3}$;
iii. $T_{Q_1} * (T_{Q_2} \circ T_{Q_3}) = (T_{Q_1} * T_{Q_2}) \circ (T_{Q_1} * T_{Q_3})$.

Proof. Proofs (i) and (ii) are similar to (iii), hence omitted. (iii) Let us prove that $T_{Q_1} * (T_{Q_2} \circ T_{Q_3}) = (T_{Q_1} * T_{Q_2}) \circ (T_{Q_1} * T_{Q_3})$ for $* = \overline{\sqcap}$ and $\circ = \sqcup$.

We consider $T_{Q_1} \,\overline{\sqcap}\, (T_{Q_2} \sqcup T_{Q_3})$. Suppose that $T_{Q_2} \sqcup T_{Q_3} = T_R$, where for all $p_j \in \mathcal{P}$:

$$t_R(p_j) = t_{Q_2 \sqcup Q_3}(p_j) = \left\{ x_i^{\langle \lambda^i_{t_R(p_j)} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_R(p_j)} = \lambda^i_{t_{(Q_2 \sqcup Q_3)(p_j)}} \in \left\{0, \frac{1}{2}, 1\right\} \right\},$$

such that:

$$\lambda^i_{t_{R(p_j)}} = \lambda^i_{t_{(Q_2 \sqcup Q_3)(p_j)}} = \max \left\{ \lambda^i_{t_{Q_2(p_j)}}, \lambda^i_{t_{Q_3(p_j)}} \right\}. \tag{4}$$

Assume that $T_{Q_1} \,\overline{\sqcap}\, T_R = T_S$, where for all $p_j \in \mathcal{P}$:

$$t_S(p_j) = t_{Q_1 \overline{\sqcap} R}(p_j) = \left\{ x_i^{\langle \lambda^i_{t_S(p_j)} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_S(p_j)} = \lambda^i_{t_{(Q_1 \overline{\sqcap} R)(p_j)}} \in \left\{0, \frac{1}{2}, 1\right\} \right\},$$

such that:

$$\lambda^i_{t_{S(p_j)}} = \lambda^i_{t_{(Q_1 \overline{\sqcap} R)(p_j)}} = \min\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{R(p_j)}}\}$$

$$= \min\{\lambda^i_{t_{Q_1(p_j)}}, \max\{\lambda^i_{t_{Q_2(p_j)}}, \lambda^i_{t_{Q_3(p_j)}}\}\}. \tag{5}$$

Now, we consider $(T_{Q_1} \bar{\sqcap} T_{Q_2}) \sqcup (T_{Q_1} \bar{\sqcap} T_{Q_3})$. Assume that $T_{Q_1} \bar{\sqcap} T_{Q_2} = T_U$, where for all $p_j \in \mathcal{P}$:

$$t_U(p_j) = t_{Q_1 \bar{\sqcap} Q_2}(p_j) = \left\{ x_i^{\langle \lambda^i_{t_U(p_j)} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_U(p_j)} = \lambda^i_{t_{(Q_1 \bar{\sqcap} Q_2)(p_j)}} \in \left\{0, \frac{1}{2}, 1\right\} \right\},$$

such that:

$$\lambda^i_{t_{U(p_j)}} = \lambda^i_{t_{(Q_1 \bar{\sqcap} Q_2)(p_j)}} = \min\left\{ \lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}} \right\}. \tag{6}$$

Assume that $T_{Q_1} \bar{\sqcap} T_{Q_3} = T_V$, where for all $p_j \in \mathcal{P}$:

$$t_V(p_j) = t_{Q_1 \bar{\sqcap} Q_2}(p_j) = \left\{ x_i^{\langle \lambda^i_{t_V(p_j)} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_V(p_j)} = \lambda^i_{t_{(Q_1 \bar{\sqcap} Q_2)(p_j)}} \in \left\{0, \frac{1}{2}, 1\right\} \right\},$$

such that:

$$\lambda^i_{t_{V(p_j)}} = \lambda^i_{t_{(Q_1 \bar{\sqcap} Q_2)(p_j)}} = \min\left\{ \lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}} \right\}. \tag{7}$$

Suppose that $T_U \sqcup T_V = T_W$, where for all $p_j \in \mathcal{P}$:

$$t_W(p_j) = t_{U \sqcup V}(p_j) = \left\{ x_i^{\langle \lambda^i_{t_W(p_j)} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_W(p_j)} = \lambda^i_{t_{(U \sqcup V)(p_j)}} \in \left\{0, \frac{1}{2}, 1\right\} \right\},$$

such that:

$$\lambda^i_{t_{W(p_j)}} = \lambda^i_{t_{(U \sqcup V)(p_j)}} = \max\left\{ \lambda^i_{t_{U(p_j)}}, \lambda^i_{t_{W(p_j)}} \right\} = \max\left\{ \min\left\{ \lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}} \right\}, \min\left\{ \lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_3(p_j)}} \right\} \right\}. \tag{8}$$

Since $\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}}, \lambda^i_{t_{Q_3(p_j)}} \in \{0, \frac{1}{2}, 1\}$, we have:

$$\lambda^i_{t_{S(p_j)}} = \lambda^i_{t_{W(p_j)}},$$

for all $p_j \in \mathcal{P}$ (by Eqs. (5) and (8). Therefore, it can be concluded that T_S and T_W are indeed the same set-valued mappings, and $T_{Q_1} \bar{\sqcap} (T_{Q_2} \sqcup T_{Q_3}) = (T_{Q_1} \bar{\sqcap} T_{Q_2}) \sqcup (T_{Q_1} \bar{\sqcap} T_{Q_3})$.

Other cases can be proved in a similar way.□

Proposition 3.6. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then, the following De Morgan's rules are held:

i. $\overline{(T_{Q_1} \bar{\sqcap} T_{Q_2})} = \overline{T_{Q_1}} \sqcup \overline{T_{Q_2}}$;

ii. $\overline{(T_{Q_1} \sqcup T_{Q_2})} = \overline{T_{Q_1}} \bar{\sqcap} \overline{T_{Q_2}}$.

Proof. (i) Since $1 - \min\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}}\} = \max\{1 - \lambda^i_{t_{Q_1(p_j)}}, 1 - \lambda^i_{t_{Q_2(p_j)}}\}$ for $\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}} \in \{0, \frac{1}{2}, 1\}$, we can say that $\overline{(T_{Q_1} \bar{\sqcap} T_{Q_2})} = \overline{T_{Q_1}} \sqcup \overline{T_{Q_2}}$. (ii) It is similar to the proof of (i).□

Definition 3.7. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then, the difference between T_{Q_1} and T_{Q_2}, denoted by $T_{Q_1} \underline{\setminus}\, T_{Q_2}$, is defined by the mapping $t_{Q_1 \underline{\setminus} Q_2} : \mathcal{P} \to \mathfrak{X}^{(\{0, \frac{1}{2}, 1\})}$ such that:

$$t_{Q_1 \underline{\setminus} Q_2}(p_j) = \left\{ x_i^{\langle \lambda^i_{t_{(Q_1 \underline{\setminus} Q_2)(p_j)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{(Q_1 \underline{\setminus} Q_2)(p_j)}} \in \{0, \frac{1}{2}, 1\} \right\},$$

for all $p_j \in \mathcal{P}$, where:

$$\lambda^i_{t_{(Q_1 \underline{\setminus} Q_2)(p_j)}} = \min\left\{ \lambda^i_{t_{Q_1(p_j)}}, 1 - \lambda^i_{t_{Q_2(p_j)}} \right\}. \tag{9}$$

Definition 3.8. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then, the symmetric difference between T_{Q_1} and T_{Q_2}, denoted by $T_{Q_1} \underline{\triangle} T_{Q_2}$, is defined by the mapping $t_{Q_1 \underline{\triangle} Q_2} : \mathcal{P} \to \mathfrak{X}^{(\{0, \frac{1}{2}, 1\})}$ such that:

$$t_{Q_1 \underline{\triangle} Q_2}(p_j) = \left\{ x_i^{\langle \lambda^i_{t_{(Q_1 \underline{\triangle} Q_2)(p_j)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{(Q_1 \underline{\triangle} Q_2)(p_j)}} \in \left\{0, \frac{1}{2}, 1\right\} \right\}$$

for all $p_j \in \mathcal{P}$, where:

$$\lambda^i_{t_{(Q_1 \underline{\triangle} Q_2)(p_j)}} = \min\left\{ \max\left\{ \lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}} \right\}, 1 - \min\left\{ \lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}} \right\} \right\}. \tag{10}$$

Example 3.6. Consider three-valued soft sets T_{Q_1} and T_{Q_2} in Tables 2 and 4, respectively. Therefore, the difference and symmetric difference between T_{Q_1} and T_{Q_2} can be measured as shown in Tables 6 and 7, respectively.

Proposition 3.7. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$.

i. $T_{Q_1} \underline{\setminus}\, T_{Q_2} = T_{Q_1} \bar{\sqcap} \overline{T_{Q_2}}$;

ii. $T_{Q_1} \underline{\triangle}\, T_{Q_2} = (T_{Q_1} \underline{\setminus}\, T_{Q_2}) \sqcup (T_{Q_2} \underline{\setminus}\, T_{Q_1})$.

Proof.

Table 6. The tabular form of $T_{Q_1} \underline{\setminus} T_{Q_2}$.

$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4
x_1	$\frac{1}{2}$	0	0	0
x_2	0	1	$\frac{1}{2}$	0
x_3	$\frac{1}{2}$	$\frac{1}{2}$	0	0
x_4	0	1	1	0

Table 7. The tabular form of $T_{Q_1} \underline{\triangle} T_{Q_2}$.

$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4
x_1	$\frac{1}{2}$	1	0	0
x_2	0	1	$\frac{1}{2}$	0
x_3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
x_4	$\frac{1}{2}$	1	1	1

(i) Assume that $T_{Q_1} \underline{\setminus} T_{Q_2} = T_R$. According to Eq. (9), for all $p_j \in \mathcal{P}$, we have:

$$\lambda^i_{t_R}(p_j) = \min\left\{\lambda^i_{t_{Q_1(p_j)}}, 1 - \lambda^i_{t_{Q_2(p_j)}}\right\}. \quad (11)$$

On the contrary, assume that $T_{Q_1} \overline{\sqcap} \overline{T_{Q_2}} = T_S$. Then, based on Definition 3.5, we have:

$$\lambda^i_{t_S}(p_j) = \min\left\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{\overline{t_{Q_2(p_j)}}}\right\} = \min\left\{\lambda^i_{t_{Q_1(p_j)}}, 1 - \lambda^i_{t_{Q_2(p_j)}}\right\}. \quad (12)$$

By Eqs. (11) and (12), we prove that the above equality is achieved.

(ii) Assume $T_{Q_1} \underline{\triangle} T_{Q_2} = T_U$. Then, by Eq. (10), we have for all $p_j \in \mathcal{P}$:

$$\lambda^i_{t_U}(p_j) = \min\left\{\max\left\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}}\right\}, 1 - \min\left\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}}\right\}\right\}. \quad (13)$$

With the consideration of the right side of equality, we have $T_{Q_1} \underline{\setminus} T_{Q_2} = T_V$ and $T_{Q_2} \underline{\setminus} T_{Q_1} = T_W$, where:

$$\lambda^i_{t_V}(p_j) = \min\left\{\lambda^i_{t_{Q_1(p_j)}}, 1 - \lambda^i_{t_{Q_2(p_j)}}\right\}, \quad (14)$$

and:

$$\lambda^i_{t_W}(p_j) = \min\left\{\lambda^i_{t_{Q_2(p_j)}}, 1 - \lambda^i_{t_{Q_1(p_j)}}\right\}. \quad (15)$$

For $T_Z = T_V \overline{\sqcap} T_W$, from Definition 3.6, we obtain that:

$$\lambda^i_{t_Z}(p_j) = \max\left\{\lambda^i_{t_{V(p_j)}}, \lambda^i_{t_{W(p_j)}}\right\} = \max\left\{\min\left\{\lambda^i_{t_{Q_1(p_j)}}, 1 - \lambda^i_{t_{Q_2(p_j)}}\right\}, \min\left\{\lambda^i_{t_{Q_2(p_j)}}, 1 - \lambda^i_{t_{Q_1(p_j)}}\right\}\right\}. \quad (16)$$

We know that $\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}} \in \{0, \frac{1}{2}, 1\}$ for all $p_j \in \mathcal{P}$, and $\lambda^i_{t_U}(p_j) = \lambda^i_{t_Z}(p_j)$. This completes the proof.□

Definition 3.9. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then, the And-product of T_{Q_1} and T_{Q_2}, denoted by $T_{Q_1} \overline{\wedge} T_{Q_2}$, is defined by the mapping $t_{Q_1 \overline{\wedge} Q_2} : \mathcal{P} \times \mathcal{P} \to \mathfrak{X}^{(\{0, \frac{1}{2}, 1\})}$ such that:

$$t_{Q_1 \overline{\wedge} Q_2}(p_j, p_k) = \left\{x_i^{\langle \lambda^i_{t_{(Q_1 \overline{\wedge} Q_2)(p_j, p_k)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{(Q_1 \overline{\wedge} Q_2)(p_j, p_k)}} \in \left\{0, \frac{1}{2}, 1\right\}\right\} \text{ for all } (p_j, p_k) \in \mathcal{P} \times \mathcal{P},$$

where:

$$\lambda^i_{t_{(Q_1 \overline{\wedge} Q_2)(p_j, p_k)}} = \min\left\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_k)}}\right\}. \quad (17)$$

Definition 3.10. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then, the Or-product of T_{Q_1} and T_{Q_2}, denoted by $T_{Q_1} \underline{\vee} T_{Q_2}$, is defined by the mapping $t_{Q_1 \underline{\vee} Q_2} : \mathcal{P} \times \mathcal{P} \to \mathfrak{X}^{(\{0, \frac{1}{2}, 1\})}$ such that:

$$t_{Q_1 \underline{\vee} Q_2}(p_j, p_k) = \left\{x_i^{\langle \lambda^i_{t_{(Q_1 \underline{\vee} Q_2)(p_j, p_k)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{(Q_1 \underline{\vee} Q_2)(p_j, p_k)}} \in \left\{0, \frac{1}{2}, 1\right\}\right\}$$

for all $(p_j, p_k) \in \mathcal{P} \times \mathcal{P}$, where:

$$\lambda^i_{t_{(Q_1 \underline{\vee} Q_2)(p_j, p_k)}} = \max\left\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_k)}}\right\}. \quad (18)$$

Proposition 3.8. Let $T_{Q_1}, T_{Q_2}, T_{Q_3} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. For all $*, \circ \in \{\overline{\wedge}, \underline{\vee}\}$:

i. $T_{Q_1} * (T_{Q_2} * T_{Q_3}) = (T_{Q_1} * T_{Q_2}) * T_{Q_3}$;
ii. $T_{Q_1} * (T_{Q_2} \circ T_{Q_3}) = (T_{Q_1} * T_{Q_2}) \circ (T_{Q_1} * T_{Q_3})$;
iii. $(T_{Q_1} \star T_{Q_2}) \circ T_{Q_3} = (T_{Q_1} \circ T_{Q_3}) \star (T_{Q_2} \circ T_{Q_3})$.

Proof. They can be shown in a similar way to the proofs of Proposition 3.5.□

Proposition 3.9. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then, the following De Morgan's laws are held:

i. $\overline{(T_{Q_1} \barwedge T_{Q_2})} = \overline{T_{Q_1}} \veebar \overline{T_{Q_2}}$;

ii. $\overline{(T_{Q_1} \veebar T_{Q_2})} = \overline{T_{Q_1}} \barwedge \overline{T_{Q_2}}$.

Proof. The proofs are similar to those of Proposition 3.6.□

4. Choice value of an object in three-valued soft set(s)

In 2002, Maji et al. [32] defined the choice value of an object in a soft set and concluded that this indication could be used to prioritize the objects in the soft set during the decision-making process. In addition, the idea of choice value initiated by Maji et al. was used for the three-valued soft sets.

In this part, it is taken $J = \{1, 2, ..., |\mathcal{P}|\}$ where $|\mathcal{P}|$ denotes the cardinality of the parameter set $\mathcal{P}$.

Definition 4.1. Let $\mathfrak{X}$ be a set of alternatives (objects).

1. The choice value of an object $x_i \in \mathfrak{X}$ in the three-valued soft set T_Q is defined and denoted by:

$$\alpha_i = \sum_{j\in J}(x_{ij})^{\xi}, \tag{19}$$

where x_{ij} for all i, j are the entries in the table of the three-valued soft set T_Q. Further, the arbitrary number $\xi \in \mathbb{R}^+$ represents the overall impact coefficient of "undetermined" on the choice value.

2. The choice value of an object $x_i \in \mathfrak{X}$ in the three-valued soft sets T_{Q_k} for $k = 1, 2, ..., s$ is defined and denoted by:

$$\alpha_i = \frac{\sum_{j\in J}((x_{ij}^{\sqcap})^{\xi} + (x_{ij}^{\sqcup})^{\xi})}{2}, \tag{20}$$

where $x_{ij}^{\sqcap}$ and $x_{ij}^{\sqcup}$ for all i, j are the entries in the tables of three-valued soft sets $\overline{\sqcap}_{k=1}^{s} T_{Q_k}$ and $\sqcup_{k=1}^{s} T_{Q_k}$, respectively. Also, the arbitrary number $\xi \in \mathbb{R}^+$ is the overall impact coefficient of "undetermined" on the choice value.

Remark: It is clear that for $x_{ij} = \frac{1}{2}$, $(x_{ij})^{\xi} \to 1$ when $\xi \to 0$ and $(x_{ij})^{\xi} \to 0$ when $\xi \to +\infty$. Take Figure 2 as an example. In case of minor modification, it is more appropriate to consider $0 < \xi < 1$. However, if the modification is major, and should consider $\xi \in (1, +\infty)$.

Algorithm 1: Selection

Step 1. Choose feasible subsets Q_k $(k = 1, 2, ..., s)$ of the parameter set $\mathcal{P}$;

Step 2. Create the three-valued soft sets T_{Q_k} for parameter subsets Q_k $(k = 1, 2, ..., s)$;

Step 3. Specify the overall impact coefficient of "undetermined" on the choice value, i.e., $\xi \in \mathbb{R}^+$;

Step 4.

- If $k > 1$, obtain the intersection and union of three-valued soft sets T_{Q_k} $(k = 1, 2, ..., s)$;
- If $k = 1$, skip to Step 4.

Step 5. Calculate α_i for all i's;

Step 6. Find l, for which $\alpha_l = \max\ \alpha_i$.

Then, x_l is the optimal choice object. If l has more than one value, any one of them could be chosen.

Example 4.1. As an implementation of Algorithm 1, we attempt to solve our numerical problem in Example 3.3. According to Tables 8 and 9, we have max $\alpha_i = \alpha_4$ for $\xi = 1, \frac{1}{4}, \frac{3}{2}, \sqrt{13}, 10$; then, the investment x_4 is an

Table 8. The choice value for three valued soft set T_{Q_1} in Example 3.3.

					$\alpha_i = \sum_j (x_{ij})^{\xi}$				
$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4	$\xi = 1$	$\xi = \frac{1}{4}$	$\xi = \frac{3}{2}$	$\xi = \sqrt{13}$	$\xi = 10$
x_1	1	0	1	0	2	2	2	2	2
x_2	0	1	$\frac{1}{2}$	0	1.5	1.8408	1.3535	1.0824	1.0009
x_3	1	$\frac{1}{2}$	$\frac{1}{2}$	0	2	2.6816	1.707	1.1648	1.0018
x_4	$\frac{1}{2}$	1	1	0	2.5	2.8408	2.3535	2.0824	2.0009

Table 9. The ranking preference order of objects for three valued soft set T_{Q_1} in Example 3.3.

	Ranking order of choice value α_i	ranking preference order of x_i	max α_i	Optimal choice object
$\xi = 1$	$\alpha_4 > \alpha_1 = \alpha_3 > \alpha_2$	$x_4 \succ x_1 \approx x_3 \succ x_2$	α_4	x_4
$\xi = \frac{1}{4}$	$\alpha_4 > \alpha_3 > \alpha_1 > \alpha_2$	$x_4 \succ x_3 \succ x_1 \succ x_2$	α_4	x_4
$\xi = \frac{3}{2}$	$\alpha_4 > \alpha_1 > \alpha_3 > \alpha_2$	$x_4 \succ x_1 \succ x_3 \succ x_2$	α_4	x_4
$\xi = \sqrt{13}$	$\alpha_4 > \alpha_1 > \alpha_3 > \alpha_2$	$x_4 \succ x_1 \succ x_3 \succ x_2$	α_4	x_4
$\xi = 10$	$\alpha_4 > \alpha_1 > \alpha_3 > \alpha_2$	$x_4 \succ x_1 \succ x_3 \succ x_2$	α_4	x_4

optimal choice to invest some money.

Example 4.2. We consider three-valued soft sets T_{Q_1} in Example 3.3 and T_{Q_2} in Example 3.5. Then, we should make a common decision based on the data in these two three-valued soft sets for $\xi = 2$. We obtain the intersection and union of T_{Q_1} and T_{Q_2} as in Tables 10 and 11, respectively.

According to Tables 10 and 11, we have $\max \alpha_i = \alpha_1 = \alpha_4 = 2.125$. It can be concluded that any of the investments x_1 and x_4 can be an optimal choice to invest some money.

In the decision-making process, all parameters of a parameter set may not be of equal importance. In such cases, weights can be imposed on the choice parameters; in other words, there is a weight $\omega_j \in (0,1]$ corresponding to each parameter $p_j \in Q$. If $p_j \in \mathcal{P} - Q$, we know that $\omega_j = 0$. Generally, the total weight is $\sum_j \omega_j = 1$. Now, let us describe the weighted choice value of an object in the structures of (weighted) three-valued soft sets.

Definition 4.2. Let $\mathfrak{X}$ be a set of alternatives (objects). Also, T_Q is a (weighted) three-valued soft set over $\mathfrak{X}$:

1. The weighted choice value of an object $x_i \in \mathfrak{X}$ in the (weighted) three-valued soft set T_Q is defined and denoted by:

$$\alpha_i^{\omega} = \sum_{j \in J} \omega_j \times (x_{ij})^{\xi}, \tag{21}$$

   where the arbitrary number $\xi \in \mathbb{R}^+$ represents the overall impact coefficient of "undetermined" on the choice value. Also, ω_j denotes the weight corresponding to each parameter p_j in the structure of three-valued soft set T_Q.

2. The weighted choice value of an object $x_i \in \mathfrak{X}$ in the (weighted) three-valued soft sets T_{Q_k} for $k = 1, 2, ..., s$ is defined and denoted by:

Table 10. Three valued soft set $T_{Q_1} \overline{\sqcap}\, T_{Q_2}$.

$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4	$\sum_j (x_{ij}^{\sqcap})^2$
x_1	$\frac{1}{2}$	0	1	0	1.25
x_2	0	0	0	0	0
x_3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0.75
x_4	$\frac{1}{2}$	0	0	0	0.25

Table 11. Three valued soft set $T_{Q_1} \underline{\sqcup}\, T_{Q_2}$.

$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4	$\sum_j (x_{ij}^{\sqcup})^2$
x_1	1	1	1	0	3
x_2	0	1	$\frac{1}{2}$	0	1.25
x_3	1	$\frac{1}{2}$	1	0	2.25
x_4	1	1	1	1	4

$$\alpha_i^{\omega} = \frac{\sum_{j \in J} \omega_j^{ort} \times ((x_{ij}^{\sqcap})^{\xi} + (x_{ij}^{\sqcup})^{\xi})}{2}, \tag{22}$$

   where the arbitrary number $\xi \in \mathbb{R}^+$ is the overall impact coefficient of "undetermined" on the choice value. Also:

$$\omega_j^{ort} = \frac{\sum_{k=1}^{s} \omega_j^k}{s}, \tag{23}$$

   where ω_j^k indicates the weight corresponding to each parameter p_j in the structure of three-valued soft set T_{Q_k}.

Algorithm 2: Selection by imposing weights on parameters

Step 1. Choose the feasible subsets Q_k ($k = 1, 2, ..., s$) of the parameter set $\mathcal{P}$ and determine its weights (i.e., ω^k) for each subsets Q_k;

Step 2. Create the (weighted) three-valued soft sets T_{Q_k} for the parameter subsets Q_k ($k = 1, 2, ..., s$);

Step 3. Specify the overall impact coefficient of "undetermined" on the choice value, i.e., $\xi \in \mathbb{R}^+$;

Step 4.

- If $k > 1$, obtain the intersection and union of three-valued soft sets T_{Q_k} ($k = 1, 2, ..., s$), and
- If $k = 1$, skip to Step 4.

Step 5. Calculate α_i^{ω} for all i's;

Step 6. Find l for which $\alpha_l^{\omega} = \max\ \alpha_i^{\omega}$.

Then, x_l is the optimal choice object. If l has more than one value, any one of them could be chosen.

Example 4.3. the numerical problem proposed in Example 3.3 was taken into consideration. Also, the following weights were measured for the parameters of Q_1: $\omega_1^1 = 0.6$ for the parameter $p_1 = \textit{high returns}$, $\omega_2^1 = 0.3$ for the parameter $p_2 = \textit{low risk}$ and $\omega_3^1 = 0.1$ for the parameter $p_3 = \textit{high security}$. Since $p_4 \notin Q_1$, the weight of parameter $p_4 \in \mathcal{P}$ can be considered "0".

According to Table 12, we have $\max\ \alpha_i^{\omega} = \alpha_3^{\omega}$ (for $\xi = 1, \frac{1}{10}, 3$). Then, the optimal choice is x_3.

Example 4.4. Consider the weighted three-valued soft sets T_{Q_1} in Example 4.3. Also, we take the following parameter weights for T_{Q_2} in Table 4: $\omega_1^2 = \omega_4^2 = 0.2$ and $\omega_2^2 = \omega_3^2 = 0.3$. For $\xi = 1$, we obtain weighted choice values of alternatives x_i ($i = 1, 2, 3, 4$) as $\alpha_1^{\omega} = 0.65$, $\alpha_2^{\omega} = 0.2$, $\alpha_3^{\omega} = 0.6$, $\alpha_4^{\omega} = 0.6$. Since $\max\ \alpha_i^{\omega} = \alpha_1^{\omega} = 0.65$, the optimal choice is x_1.

In Algorithms 1 and 2, for each decision-maker, the overall impact coefficient (ξ) of "undetermined" on the choice value is taken the same. These two algorithms cannot be used if each decision-maker selects the impact coefficient of "undetermined" differently. To address these shortcomings, we will create new decision-making algorithms.

Table 12. The weighted choice value for three valued soft set T_{Q_1} in Example 3.3

					$\alpha_i^\omega = \sum_j \omega_j^1 \times (x_{ij})^\xi$		
$\mathfrak{X}/\mathcal{P}$	$p_1(\omega_1^1 = 0.6)$	$p_2(\omega_2^1 = 0.3)$	$p_3(\omega_3^1 = 0.1)$	$p_4(\omega_4^1 = 0)$	$\xi = 1$	$\xi = \frac{1}{10}$	$\xi = 3$
x_1	1	0	1	0	0.7	0.7	0.7
x_2	0	1	$\frac{1}{2}$	0	0.35	0.3933	0.3125
x_3	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0.8	0.9732	0.65
x_4	$\frac{1}{2}$	1	1	0	0.7	0.9598	0.475

5. Three-valued soft decision-making model based on TOPSIS

This section primarily focuses on the TOPSIS, which produces satisfactory results during decision-making. Here, this technique was rebuilt to deal with the multi-criteria-group decision-making problems based on three-valued soft sets, thus constructing a novel decision-making model.

TOPSIS is an approach to solving multi-criteria decision-making problems based on a decision maker. In 2007, Shih et al. [47] extended this method for group decision-making. The operations in the process of TOPSIS include decision matrix normalization, distance measures, and aggregation operators [47]. Generally, a decision matrix is required prior to the beginning of the process. As a result of this process, the output data are interpreted so that the ranking order of alternatives can be obtained. In summary, the TOPSIS approach is a practical and useful method for ranking and selecting a number of externally determined alternatives through distance measures. The main procedure of TOPSIS is given in a series of steps (see [48–51]).

Now, a multi-criteria group decision-making model is proposed using the TOPSIS on three-valued soft sets.

Algorithm 3: TOPSIS based three-valued soft sets

Step 1. The multi-criteria group decision-making problem is identified. In this step, decision makers (experts), alternatives, and choice parameters are determined. Suppose that $DM = \{E_k : k \in I_s = \{1, 2, ..., s\}\}$ is a set of decision makers (experts) and E_k denotes the kth decision maker (expert). Also, x_i ($i \in \mathcal{I}_m = \{1, 2, ..., m\}$) denotes the ith alternative, and p_j ($j \in \mathcal{I}_n = \{1, 2, ..., n\}$) represents the jth parameter (criterion or attribute).

Considering these data, each decision maker E_k ($k \in I_s$) create three-valued soft set T_{Q_k} and measures the weights of parameters as ω_j^k ($j \in \mathcal{I}_n$) satisfying the condition $\sum_{j=1}^n \omega_j^k = 1$.

Moreover, each decision maker E_k specifies the impact coefficient of "undetermined" in decision-making, i.e., ξ_k;

Step 2. For each decision maker E_k, the decision matrix $\mathfrak{D}^k$ is constructed and represented as follows:

$$\mathfrak{D}^k = \begin{array}{c} \\ x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{array} \begin{array}{c} \begin{array}{ccccc} p_1 & p_2 & \cdot & \cdot\ \cdot & p_n \end{array} \\ \begin{bmatrix} d_{11}^k & d_{12}^k & \cdot & \cdot\ \cdot & d_{1n}^k \\ d_{21}^k & d_{22}^k & \cdot & \cdot\ \cdot & d_{2n}^k \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ d_{m1}^k & d_{m2}^k & \cdot & \cdot\ \cdot & d_{mn}^k \end{bmatrix} \end{array} = [d_{ij}^k]_{m \times n},$$

where $d_{ij}^k = (x_{ij}^k)^{\xi_k}$ that x_{ij}^k for all i, j are the entries in the table of three-valued soft set T_{Q_k};

Step 3. After constructing the decision matrices, these are normalized (standardized).

For each decision matrix $\mathfrak{D}^k$, the normalized decision matrix $\mathfrak{R}^k$ is constructed and expressed as follows:

$$\mathfrak{R}^k = \begin{bmatrix} r_{11}^k & r_{12}^k & \cdot & \cdot\ \cdot & r_{1n}^k \\ r_{21}^k & r_{22}^k & \cdot & \cdot\ \cdot & r_{2n}^k \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ r_{m1}^k & r_{m2}^k & \cdot & \cdot\ \cdot & r_{mn}^k \end{bmatrix} = [r_{ij}^k]_{m \times n},$$

where:

$$r_{ij}^k = \begin{cases} \dfrac{d_{ij}^k}{\sqrt{\sum_{\ell=1}^m (d_{\ell j}^k)^2}}, & \text{if } d_{ij}^k \neq 0 \\ 0, & \text{if } d_{ij}^k = 0 \end{cases} \tag{24}$$

for all $k \in \mathcal{I}_s$, $i \in \mathcal{I}_m$, and $j \in \mathcal{I}_n$.

Step 4. Given different weights of parameters for each decision-maker, the weighted normalized decision matrix is calculated by multiplying the weights of evaluation parameters by values in the normalized decision matrix.

For each normalized decision matrix $\mathfrak{R}^k$, the weighted normalized decision matrix V^k is created as follows:

$$\mathrm{V}^k = \begin{bmatrix} \mathrm{v}_{11}^k & \mathrm{v}_{12}^k & \cdot & \cdot\ \cdot & \mathrm{v}_{1n}^k \\ \mathrm{v}_{21}^k & \mathrm{v}_{22}^k & \cdot & \cdot\ \cdot & \mathrm{v}_{2n}^k \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \mathrm{v}_{m1}^k & \mathrm{v}_{m2}^k & \cdot & \cdot\ \cdot & \mathrm{v}_{mn}^k \end{bmatrix} = [\mathrm{v}_{ij}]_{m \times n},$$

where:

$$\mathrm{v}_{ij}^k = w_j^k \times r_{ij}^k, \tag{25}$$

for all $k \in \mathcal{I}_s$, $i \in \mathcal{I}_m$ and $j \in \mathcal{I}_n$.

Step 5. By combining the weighted normalized decision matrices V^k $(k \in \mathcal{I}_s)$, the average weighted normalized decision matrix $\mathcal{V}$ can be obtained.

The structure of the matrix $\mathcal{V}$ is expressed as follows:

$$\mathcal{V} = \begin{bmatrix} v_{11} & v_{12} & . & . & . & v_{1n} \\ v_{21} & v_{22} & . & . & . & v_{2n} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ v_{m1} & v_{m2} & . & . & . & v_{mn} \end{bmatrix} = [v_{ij}]_{m\times n},$$

where for all $i \in \mathcal{I}_m$ and $j \in \mathcal{I}_n$:

$$v_{ij} = \mathrm{v}_{ij}^1 \oplus \mathrm{v}_{ij}^2 \oplus ... \oplus \mathrm{v}_{ij}^s. \tag{26}$$

In other words, the value v_{ij} is obtained by combining the weighted normalized decision value v_{ij}^k through an operation $\oplus$. Here, the operation $\oplus$ can offer many choices: arithmetic mean, geometric mean, harmonic mean and their modification.

Note: In this study, we will take the arithmetic mean of all individual measures.

Step 6. The positive and negative ideal solutions $\mathcal{V}^\top$ and $\mathcal{V}^\perp$ are determined using the average weighted normalized decision matrix $\mathcal{V}$.

In the TOPSIS approach, the parameters (criteria or attributes) are evaluated in terms of benefit (cf. Example 3.3) and cost (cf. Example 3.1). Suppose that $\mathcal{J}_1$ and $\mathcal{J}_2$ are the sets of benefit and cost parameters, respectively, where $\mathcal{J}_1 \cap \mathcal{J}_2 = \phi$ and $\mathcal{J}_1 \cup \mathcal{J}_2 = \{1, 2, ..., n\}$. $\mathcal{V}^\top$ and V_L are described as follows:

- $\mathcal{V}^\top$ is the set which shows that the most suitable alternative for each parameter may be preferred (PIS). This set is obtaibed and shown in the following:

$$\begin{aligned}\mathcal{V}^\top &= \{v_1^\top, v_2^\top, ..., v_j^\top, ..., v_n^\top\} \\ &= \{(\max_i v_{ij}^\perp : j \in \mathcal{J}_1), (\min_i v_{ij}^\perp : j \in \mathcal{J}_2), \\ & i \in \mathcal{I}_m\}. \end{aligned} \tag{27}$$

- $\mathcal{V}^\perp$ is the set showing the least preferable alternative for each parameter (NIS). This set is obtained as follows:

$$\begin{aligned}\mathcal{V}^\perp &= \{v_1^\perp, v_2^\perp, ..., v_j^\perp, ..., v_n^\perp\} \\ &= \{(\min_i v_{ij}^\perp : j \in \mathcal{J}_1), (\max_i v_{ij}^\perp : j \in \mathcal{J}_2), \\ & i \in \mathcal{I}_m\}. \end{aligned} \tag{28}$$

Step 7. The separation measurements of alternatives to the ideal solutions are obtained through the Euclidean distance formula.

The separation measurement of each alternative x_i to the positive ideal solution $\mathcal{V}^\top$ is calculated as follows:

$$\mathcal{S}_i^\top = \sqrt{\sum_{j=1}^n (v_{ij} - v_j^\top)^2}. \tag{29}$$

The separation measurement of each alternative x_i to the negative ideal solution $\mathcal{V}^\perp$ is calculated as follows:

$$\mathcal{S}_i^\perp = \sqrt{\sum_{j=1}^n (v_{ij} - v_j^\perp)^2}. \tag{30}$$

Here, $\mathcal{S}_i^\top$ and $\mathcal{S}_i^\perp$ represent the distance of the alternative x_i from PIS and NIS, respectively.

Step 8. The relative closeness of each alternative to the ideal solutions is also calculated.

The relative closeness $\mathcal{C}_i^\dagger$ of the alternatives x_i with respect to the ideal solutions can be expressed as:

$$\mathcal{C}_i^\dagger = \frac{\mathcal{S}_i^\perp}{\mathcal{S}_i^\perp + \mathcal{S}_i^\top}, \ \forall i \in \mathcal{I}_m \ \ (0 \le \mathcal{C}_i^\dagger \le 1). \tag{31}$$

Step 9. The alternatives (objects) are ranked in order of preference.

A set of alternatives x_i can be ranked according to the descending order of the values $\mathcal{C}_i^\dagger$.

To show the potential of the proposed approach, a real-life practice was suggested, adopted from Figure 4.

Example 5.1. Assume that two experts are about to determine the best mobile brand by examining the new model mobile phones presented by six different mobile phone brands. The first expert E_1 reviews each brand's mobile phone with memory of 128 GB, and the second expert E_2 reviews each brand's mobile phone with memory of 64 GB. Let $\mathfrak{X} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be a set of six different mobile phone brands. Also, the set of parameters is employed to determine the brand with the best mobile phones, which is given as $\mathcal{P} = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$ where $p_1 = optical$ image stabilization, $p_2 = quick$ charging, $p_3 = expandable\ memory$, $p_4 = waterproof$, $p_5 = autofocus$, $p_6 = cheap$, and $p_7 = fingerprint\ sensor$. Each of these experts proceeds to the decision making stage after reviewing comparisons on mobile phones on a comparison-focused website (such as "www.91mobiles.com" and "www.smartprix.com").

To deal with this problem, the steps of Algorithm 3 are followed:

Step 1. The experts E_1 and E_2 determine the

Table 13. The weights of choice parameters of experts E_1 and E_2.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	
First expert (E_1)	0.22	0.15	0.13	0.15	0.13	0.11	0.11	$\sum_{j=1}^{7}\omega_j^1 = 1$
Second expert (E_2)	0.24	0.12	0.12	0.16	0.14	0.1	0.12	$\sum_{j=1}^{7}\omega_j^2 = 1$

Table 14. Three valued soft sets of experts E_1 and E_2.

Decision makers:	**First expert (E_1)**							**Second expert (E_2)**						
$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_1	p_2	p_3	p_4	p_5	p_6	p_7
x_1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	1	0
x_2	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	1	1	1	1
x_3	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	0	1	1	0	1
x_4	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	$\frac{1}{2}$	1	0	0	1	$\frac{1}{2}$
x_5	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	0	1	0	1	1
x_6	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	1	1	1	1	1	1	0	1	1
Impact coefficient ξ_k:	$\xi_1 = 1.5$							$\xi_2 = 2$						

parameter sets as $Q_1 = Q_2 = \mathcal{P}$, respectively. In addition, they determine the weights of their choice parameters as ω_j^k for all $j = 1, 2, ..., 7$ and $k = 1, 2$. (see Table 13). The experts collect data about each brand's mobile phone x_i $(i = 1, 2, ..., 6)$ for each attribute p_j and create three-valued soft sets in Table 14.

Step 2. According to Table 14, the decision matrices $\mathfrak{D}^k$ $(k = 1, 2)$ are constructed in the equations shown in Box VI.

Step 3. For each decision matrix $\mathfrak{D}^k$ $(k = 1, 2)$, the normalized decision matrix $\mathfrak{R}^k$ $(k = 1, 2)$ is constructed in the equations shown in Box VII.

Step 4. For each normalized decision matrix $\mathfrak{R}^k$ $(k = 1, 2)$, the weighted normalized decision matrix V^k $(k = 1, 2)$ is formed by the equations shown in Box VIII.

Step 5. Then, the average weighted normalized decision matrix is constructed by the equation shown in Box IX, where the operation $\oplus$ represents the arithmetic mean.

Step 6. The positive and negative ideal solutions $\mathcal{V}^\top$ and $\mathcal{V}^\perp$ are determined as follows:

$$\mathcal{V}^\top = \{v_1^\top = 0.1524, v_2^\top = 0.094, v_3^\top = 0.0764, v_4^\top = 0.0763, v_5^\top = 0.1017, v_6^\top = 0.0498, v_7^\top = 0.0568\},$$

$$\mathfrak{D}^1 = \begin{bmatrix} 1 & 0 & 0 & 0.3535 & 0.3535 & 1 & 0 \\ 1 & 0.3535 & 0.3535 & 1 & 1 & 1 & 0.3535 \\ 0.3535 & 0 & 0.3535 & 1 & 0.3535 & 0 & 1 \\ 0.3535 & 0.3535 & 1 & 0.3535 & 0.3535 & 1 & 1 \\ 0 & 0.3535 & 0.3535 & 1 & 0.3535 & 0 & 1 \\ 0.3535 & 0.3535 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} = [d_{ij}^1]_{6\times 7},$$

$$\mathfrak{D}^2 = \begin{bmatrix} 0.25 & 0.25 & 1 & 0 & 0.25 & 1 & 0 \\ 1 & 0 & 0.25 & 1 & 1 & 1 & 1 \\ 0.25 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0.25 & 1 & 0 & 0 & 1 & 0.25 \\ 0.25 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} = [d_{ij}^2]_{6\times 7}.$$

Box VI

$$\mathfrak{R}^1 = \begin{bmatrix} 0.6489 & 0 & 0 & 0.1714 & 0.2886 & 0.5 & 0 \\ 0.6489 & 0.5001 & 0.2293 & 0.485 & 0.8166 & 0.5 & 0.174 \\ 0.2293 & 0 & 0.2293 & 0.485 & 0.2886 & 0 & 0.4923 \\ 0.2293 & 0.5001 & 0.6489 & 0.1714 & 0.2886 & 0.5 & 0.4923 \\ 0 & 0.5001 & 0.2293 & 0.485 & 0.2886 & 0 & 0.4923 \\ 0.2293 & 0.5001 & 0.2293 & 0.485 & 0 & 0.5 & 0.4923 \end{bmatrix},$$

$$\mathfrak{R}^2 = \begin{bmatrix} 0.169 & 0.2357 & 0.5714 & 0 & 0.0174 & 0.4472 & 0 \\ 0.6761 & 0 & 0.1428 & 0.5 & 0.6963 & 0.4472 & 0.4961 \\ 0.169 & 0 & 0 & 0.5 & 0.6963 & 0 & 0.4961 \\ 0 & 0.2357 & 0.5714 & 0 & 0 & 0.4472 & 0.124 \\ 0.169 & 0 & 0 & 0.5 & 0 & 0.4472 & 0.4961 \\ 0.6761 & 0.9428 & 0.5714 & 0.5 & 0 & 0.4472 & 0.4961 \end{bmatrix}.$$

Box VII

$$\mathrm{V}^1 = \begin{bmatrix} 0.1427 & 0 & 0 & 0.0257 & 0.0375 & 0.055 & 0 \\ 0.1427 & 0.075 & 0.0298 & 0.0727 & 0.1061 & 0.055 & 0.0191 \\ 0.0504 & 0 & 0.0298 & 0.0727 & 0.0375 & 0 & 0.0541 \\ 0.0504 & 0.075 & 0.0843 & 0.0257 & 0.0375 & 0.055 & 0.0541 \\ 0 & 0.075 & 0.0298 & 0.0727 & 0.0375 & 0 & 0.0541 \\ 0.0504 & 0.075 & 0.0298 & 0.0727 & 0 & 0.055 & 0.0541 \end{bmatrix},$$

$$\mathrm{V}^2 = \begin{bmatrix} 0.0405 & 0.0282 & 0.0685 & 0 & 0.0243 & 0.0447 & 0 \\ 0.1622 & 0 & 0.0171 & 0.08 & 0.0974 & 0.0447 & 0.0595 \\ 0.0405 & 0 & 0 & 0.08 & 0.0974 & 0 & 0.0595 \\ 0 & 0.0282 & 0.0685 & 0 & 0 & 0.0447 & 0.0148 \\ 0.0405 & 0 & 0 & 0.08 & 0 & 0.0447 & 0.0595 \\ 0.1622 & 0.1131 & 0.0685 & 0.08 & 0 & 0.0447 & 0.0595 \end{bmatrix}.$$

Box VIII

and:

$$\mathcal{V}^{\perp} = \{v_1^{\perp} = 0.0202, v_2^{\perp} = 0, v_3^{\perp} = 0.0149,$$

$$v_4^{\perp} = 0.0128, v_5^{\perp} = 0, v_6^{\perp} = 0, v_7^{\perp} = 0\}.$$

Step 7. The separation measurements $\mathcal{S}_i^{\top}$ and $\mathcal{S}_i^{\perp}$ of each alternative x_i to the ideal solutions are given in Table 15.

Step 8. The relative closeness $\mathcal{C}_i^{\dagger}$ of each alternative to the ideal solutions is calculated as follows:

$$\mathcal{C}_1^{\dagger} = 0.3767, \qquad \mathcal{C}_2^{\dagger} = 0.7095,$$

$$\mathcal{C}_3^{\dagger} = 0.399, \qquad \mathcal{C}_4^{\dagger} = 0.3706,$$

$$\mathcal{C}_5^{\dagger} = 0.3513, \qquad \mathcal{C}_6^{\dagger} = 0.589.$$

Step 9. According to the descending order of

$$\mathcal{V} = \mathrm{V}^1 \oplus \mathrm{V}^2 = \begin{bmatrix} 0.0916 & 0.0141 & 0.0342 & 0.0128 & 0.0309 & 0.0498 & 0 \\ 0.1524 & 0.0375 & 0.0234 & 0.0763 & 0.1017 & 0.0498 & 0.0393 \\ 0.0454 & 0 & 0.0149 & 0.0763 & 0.0674 & 0 & 0.0568 \\ 0.0252 & 0.0516 & 0.0764 & 0.0128 & 0.0187 & 0.0498 & 0.0344 \\ 0.0202 & 0.0375 & 0.0149 & 0.0763 & 0.0187 & 0.0223 & 0.0568 \\ 0.1063 & 0.094 & 0.0491 & 0.0763 & 0 & 0.0498 & 0.0568 \end{bmatrix} = [v_{ij}]_{6\times 7}.$$

Box IX

Table 15. Separation measurements $\mathcal{S}_i^{\top}$ and $\mathcal{S}_i^{\perp}$.

	x_1	x_2	x_3	x_4	x_5	x_6
$\mathcal{S}_i^{\top}$	0.1542	0.0787	0.167	0.1705	0.178	0.1144
$\mathcal{S}_i^{\perp}$	0.0932	0.1923	0.1109	0.1004	0.0964	0.164

the values $\mathcal{C}_i^{\dagger}$, the ranking order of alternatives is obtained below:

$$x_2 \succ x_6 \succ x_3 \succ x_1 \succ x_4 \succ x_5.$$

Then, it can be argued that x_2 is the best mobile phone brand according to the data presented by experts.

6. Three-valued soft decision-making model based on ELECTRE

This section introduces a modified version of ELECTRE technique ("ELimination Et Choix Traduisant la REalité" or "Elimination and Choice Expressing Reality"), which is generally intended to output choosing, sorting, and ranking, to deal with the multi-criteria group decision-making problems based on three-valued soft sets.

As was first applied in 1965, the ELECTRE method was employed to choose the best alternative(s) from a given set of alternatives and it was applied to three fundamental problems:

Choosing: Selecting a restricted number of the most interesting potential alternatives, as small as possible which will justify elimination of the others.

Sorting: Assigning each potential alternative to one of the categories a family previously described; the categories are ordered from the worst to best .

Ranking: Ordering alternatives from the best to worst with the possibility of ties.
The main procedure of ELECTRE is described in a series of steps (see [52–56].

Now, a multi-criteria group decision making model is constructed on three-valued soft sets using the modified ELECTRE technique.

Algorithm 4: ELECTRE based three-valued soft sets

Step 1. Describe the multi-criteria group-decision-making problem (the same as Step 1 in Algorithm 3).

Step 2. For each decision maker E_k, the decision matrix $\mathfrak{D}^k$ is constructed (the same as Step 2 in Algorithm 3).

Step 3. For each decision matrix $\mathfrak{D}^k$, the normalized decision matrix $\mathfrak{R}^k$ is constructed (the same as Step 3 in Algorithm 3).

Step 4. For each normalized decision matrix $\mathfrak{R}^k$, the weighted normalized decision matrix V^k is created (the same as Step 4 in Algorithm 3).

Step 5. After combining the weighted normalized decision matrices V^k $(k \in \mathcal{I}_s)$, the average weighted normalized decision matrix $\mathcal{V}$ is formed (the same as Step 5 in Algorithm 3).

Step 6. The concordance sets and discordance sets are determined. The concordance set is composed of the index of all parameters for which the alternative x_τ is preferred over the alternative x_κ. This set can be described as follows.

For $\tau, \kappa \in \mathcal{I}_m$ and $\tau \neq \kappa$ (note that an alternative is not compared to itself):

$$\mathcal{J}_{\tau\kappa}^{+} = \{j : \ v_{\tau j} \geq v_{\kappa j}\}. \tag{32}$$

The discordance set contains the index of all parameters for which the alternative x_τ is worse than the alternative x_κ. This set can be described as follows.

For $\tau, \kappa \in \mathcal{I}_m$ and $\tau \neq \kappa$:

$$\mathcal{J}_{\tau\kappa}^{-} = \{j : \ v_{\tau j} < v_{\kappa j}\}. \tag{33}$$

In other words, this set can be considered as the complement of the concordance set $\mathcal{J}_{\tau\kappa}^{+}$, i.e., $\mathcal{J}_{\tau\kappa}^{-} = \mathcal{J} \setminus \mathcal{J}_{\tau\kappa}^{+}$ where $\mathcal{J} = \{j : \ p_j \in \mathcal{P}\}$.

Step 7. The concordance matrix and discordance matrix are generated by employing the sets of concordance and discordance, respectively.

The concordance matrix can be expressed as follows:

$$\mathcal{A} = \begin{bmatrix} - & \cdot & \cdot & \cdot & a_{1\kappa} & \cdot & \cdot & \cdot & a_{1m} \\ \cdot & \cdot & & & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{\tau 1} & \cdot & \cdot & \cdot & a_{\tau\kappa} & \cdot & \cdot & \cdot & a_{\tau m} \\ \cdot & \cdot & & & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \cdot & \cdot & \cdot & a_{m\kappa} & \cdot & \cdot & \cdot & - \end{bmatrix}$$

$$= [a_{\tau\kappa}]_{m\times m},$$

where:

$$a_{\tau\kappa}=\frac{\sum_{j\in\mathcal{J}^{+}_{\tau\kappa}}\sum_{k=1}^{s}\omega_j^k}{\sum_{j\in\mathcal{J}}\sum_{k=1}^{s}\omega_j^k},\quad (0\leq a_{\tau\kappa}\leq 1), \tag{34}$$

for all $\tau,\kappa\in\mathcal{I}_m$. In other words, each component of concordance matrix is found as a summation of the (standardized) weights of all parameters corresponding to the indices in the concordance set $\mathcal{J}^{+}$.

The discordance matrix can be expressed as follows:

$$\mathcal{B}=\begin{bmatrix} - & \cdots & b_{1\kappa} & \cdots & b_{1m} \\ \vdots & & \vdots & & \vdots \\ b_{\tau 1} & \cdots & b_{\tau\kappa} & \cdots & b_{\tau m} \\ \vdots & & \vdots & & \vdots \\ b_{m1} & \cdots & b_{m\kappa} & \cdots & - \end{bmatrix}$$

$$=[b_{\tau\kappa}]_{m\times m},$$

where:

$$b_{\tau\kappa}=\frac{\sum_{j\in\mathcal{J}^{-}_{\tau\kappa}}|\mathrm{v}_{\tau j}-\mathrm{v}_{\kappa j}|}{\sum_{j\in\mathcal{J}}|\mathrm{v}_{\tau j}-\mathrm{v}_{\kappa j}|},\quad (0\leq b_{\tau\kappa}\leq 1), \tag{35}$$

for all $\tau,\kappa\in\mathcal{I}_m$.

Step 8. The concordance threshold $\underline{\mathcal{A}}$ and discordance threshold $\underline{\mathcal{B}}$ are found.

The concordance threshold is calculated as follows:

$$\underline{\mathcal{A}}=\frac{\sum\limits_{\tau=1}^{m}\sum\limits_{\kappa=1}^{m}a_{\tau\kappa}}{m(m-1)},\quad (0\leq\underline{\mathcal{A}}\leq 1), \tag{36}$$

and the discordance threshold is calculated below:

$$\underline{\mathcal{B}}=\frac{\sum\limits_{\tau=1}^{m}\sum\limits_{\kappa=1}^{m}b_{\tau\kappa}}{m(m-1)},\quad (0\leq\underline{\mathcal{B}}\leq 1). \tag{37}$$

Step 9. The effective concordance matrix $\mathcal{F}$ and effective discordance matrix $\mathcal{G}$ are created.

The effective concordance matrix $\mathcal{F}$ is measured based on the concordance threshold $\underline{\mathcal{A}}$, as expressed in the following:

$$\mathcal{F}=\begin{bmatrix} - & \cdots & f_{1\kappa} & \cdots & f_{1m} \\ \vdots & & \vdots & & \vdots \\ f_{\tau 1} & \cdots & f_{\tau\kappa} & \cdots & f_{\tau m} \\ \vdots & & \vdots & & \vdots \\ f_{m1} & \cdots & f_{m\kappa} & \cdots & - \end{bmatrix}$$

$$=[f_{\tau\kappa}]_{m\times m},$$

where:

$$f_{\tau\kappa}=\begin{cases}1, & \text{if } a_{\tau\kappa}\geq\underline{\mathcal{A}}\\ 0, & \text{if } a_{\tau\kappa}<\underline{\mathcal{A}}\end{cases} \tag{38}$$

The effective discordance matrix $\mathcal{G}$ is measured based on the discordance threshold $\underline{\mathcal{B}}$, as expressed in the following:

$$\mathcal{G}=\begin{bmatrix} - & \cdots & g_{1\kappa} & \cdots & g_{1m} \\ \vdots & & \vdots & & \vdots \\ g_{\tau 1} & \cdots & g_{\tau\kappa} & \cdots & g_{\tau m} \\ \vdots & & \vdots & & \vdots \\ g_{m1} & \cdots & g_{m\kappa} & \cdots & - \end{bmatrix}$$

$$=[g_{\tau\kappa}]_{m\times m},$$

where:

$$g_{\tau\kappa}=\begin{cases}0, & \text{if } b_{\tau\kappa}>\underline{\mathcal{B}}\\ 1, & \text{if } b_{\tau\kappa}\leq\underline{\mathcal{B}}\end{cases} \tag{39}$$

Step 10. The aggregated outranking matrix $\mathcal{H}$ is constructed. Then, the aggregated outranking matrix $\mathcal{H}$ is established by merging the effective concordance information with effective discordance information. The matrix $\mathcal{H}$ can be described as follows:

$$\mathcal{H}=\begin{bmatrix} - & \cdots & h_{1\kappa} & \cdots & h_{1m} \\ \vdots & & \vdots & & \vdots \\ h_{\tau 1} & \cdots & h_{\tau\kappa} & \cdots & h_{\tau m} \\ \vdots & & \vdots & & \vdots \\ h_{m1} & \cdots & h_{m\kappa} & \cdots & - \end{bmatrix}$$

$$=[h_{\tau\kappa}]_{m\times m},$$

where:

$$h_{\tau\kappa}=f_{\tau\kappa}\times g_{\tau\kappa}, \tag{40}$$

for all $\tau,\kappa\in\mathcal{I}_m$.

Step 11. The alternatives (objects) are ranked in order of preference. The components in the aggregated outranking matrix $\mathcal{H}$ are indicative of the dominance of any alternative over the other. Therefore, according to this priority, a choice priority among the alternatives is considered to rank the alternatives.

Given the aggregated outranking matrix $\mathcal{H}$, the binary relations among the alternatives may take place as one of the following three situations:

(a) $x_\tau\succ x_\kappa$ (i.e., x_τ is strictly preferred over x_κ or x_τ is dominant over x_κ) if $h_{\tau\kappa}=1$ and $h_{\kappa\tau}=0$;

(b) $x_\tau \approx x_\kappa$ (i.e., x_τ is indifferent to x_κ) if $h_{\tau\kappa} = 1$ and $h_{\kappa\tau} = 1$;

(c) $x_\tau ? x_\kappa$ (i.e., x_τ and x_κ are incomparable) if $h_{\tau\kappa} = 0$ and $h_{\kappa\tau} = 0$.

Therefore, the ranking order of alternatives can be interpreted.

For the implementation of this model, a solution that follows the multi-criteria group-decision-making problem is offered.

Example 6.1. Supplier selection is among the most important issues in the supply chain management area. In this regard, a numerical example of a supplier selection problem adopted from [57,58] was taken into account. A high-technology company that manufactures electronic products aims to evaluate and choose a materials supplier. Assume that $\mathfrak{X} = \{x_1, x_2, x_3, x_4\}$ is a set of four suppliers chosen as candidates (alternatives). A single decision maker may not be able to accurately consider all relevant aspects during decision-making. Therefore, the company's leader decides to put together a decision committee to determine a suitable supplier. A committee of three decision makers (experts) is established containing: 1) Financial expert (E_1), who evaluates alternatives in terms of cost and finance; 2) Quality control expert (E_2), who evaluates alternatives in terms of quality and safety; and 3) Engineering expert (E_3), who evaluates alternatives in terms of engineering and technical aspects.

Six, evaluation parameters are also considered: 1) performance (p_1), 2) cost control (p_2), 3) management audit (p_3), 4) service (p_4), 5) company reputation (p_5), and 6) quality (p_6). To deal with this problem, the steps of Algorithm 4 are followed:

Step 1. Decision makers (experts) E_1, E_2, and E_3 make their decisions based on the parameter subsets $Q_1 = \{p_1, p_2, p_3, p_5\}$, $Q_2 = \{p_1, p_2, p_3, p_5, p_6\}$, and $Q_3 = \{p_1, p_2, p_4, p_5, p_6\}$, respectively. In addition, they measure the weights of their choice parameters, as shown in Table 16.

Decision makers (experts) present their opinion about the truth of alternative x_i under the parameter p_j and construct Table 17.

Step 2. The (three-valued) decision matrices $\mathfrak{D}^k$ ($k = 1, 2, 3$) are constructed as follows:

$$\mathfrak{D}^1 = \begin{bmatrix} 1 & 0.25 & 0 & 0 & 1 & 0 \\ 0.25 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0.25 & 0 \\ 0.25 & 0.25 & 0.25 & 0 & 1 & 0 \end{bmatrix},$$

$$\mathfrak{D}^2 = \begin{bmatrix} 0 & 0.125 & 1 & 0 & 0.125 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0.125 & 0 \\ 0.125 & 0 & 0.125 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathfrak{D}^3 = \begin{bmatrix} 0.25 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0.25 & 0 & 0 & 0.25 & 0 \\ 1 & 0 & 0 & 0 & 0.25 & 1 \\ 1 & 1 & 0 & 1 & 0.25 & 0 \end{bmatrix}.$$

Steps 3 and 4. For the decision matrices $\mathfrak{D}^k$ ($k = 1, 2, 3$), the normalized decision matrices $\mathfrak{R}^k$ ($k = 1, 2, 3$) and weighted normalized decision matrices V^k ($k = 1, 2, 3$) are constructed similar to that in Steps 3 and 4 of Example 5.1; hence, it is omitted.

Table 16. The weights of the decision maker's choice parameters.

Decision makers/weights	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	
Engineering expert (E_1)	0.2	0.15	0.5	0	0.15	0	$\sum_{j=1}^{6} \omega_j^1 = 1$
Financial expert (E_2)	0.15	0.25	0.25	0	0.1	0.25	$\sum_{j=1}^{6} \omega_j^2 = 1$
Quality control expert (E_3)	0.2	0.2	0	0.1	0.1	0.4	$\sum_{j=1}^{6} \omega_j^3 = 1$

Table 17. The expert's three valued soft sets T_{Q_1}, T_{Q_2}, and T_{Q_3}.

Decision makers:	Financial expert (E_1)						Quality control expert (E_2)						Engineering expert (E_3)					
$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4	p_5	p_6	p_1	p_2	p_3	p_4	p_5	p_6	p_1	p_2	p_3	p_4	p_5	p_6
x_1	1	$\frac{1}{2}$	0	0	1	0	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0
x_2	$\frac{1}{2}$	1	1	0	0	0	1	0	0	0	0	1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0
x_3	0	1	1	0	$\frac{1}{2}$	0	1	1	1	0	$\frac{1}{2}$	0	1	0	0	0	$\frac{1}{2}$	1
x_4	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	1	1	1	0	1	$\frac{1}{2}$	0
Impact coefficient ξ_k:	$\xi_1 = 2$						$\xi_2 = 3$						$\xi_3 = 2$					

Table 18. Sets of concordance and discordance.

	Concordance set $(\mathcal{J}^+)$				Discordance set $(\mathcal{J})$			
	x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
x_1	–	$\{1,4,5\}$	$\{4,5\}$	$\{1,3,5\}$	–	$\{2,3,6\}$	$\{1,2,3,6\}$	$\{2,4,6\}$
x_2	$\{2,3,4,6\}$	–	$\{4\}$	$\{3,6\}$	$\{1,5\}$	–	$\{1,2,3,5,6\}$	$\{1,2,4,5\}$
x_3	$\{1,2,3,4,6\}$	$\{1,2,3,4,5,6\}$	–	$\{1,2,3,6\}$	$\{5\}$	$\emptyset$	–	$\{4,5\}$
x_4	$\{2,4,6\}$	$\{1,2,4,5,6\}$	$\{4,5\}$	–	$\{1,3,5\}$	$\{3\}$	$\{1,2,3,6\}$	–

Step 5. Then, the average weighted normalized decision matrix is:

$$\mathcal{V} = \mathrm{V}^1 \oplus \mathrm{V}^2 \oplus \mathrm{V}^3 =$$

$$\begin{bmatrix} 0.0744 & 0.0189 & 0.0586 & 0 & 0.0917 & 0 \\ 0.0509 & 0.0504 & 0.116 & 0 & 0.0333 & 0.0589 \\ 0.0816 & 0.117 & 0.1747 & 0 & 0.0656 & 0.1333 \\ 0.0665 & 0.0732 & 0.0363 & 0.0333 & 0.0681 & 0.0589 \end{bmatrix}$$

$$= [v_{ij}]_{4\times 6},$$

where the operation $\oplus$ represents the arithmetic mean.

Step 6. With the consideration of the average weighted normalized decision matrix $\mathcal{V}$, the concordance set and discordance set are determined and presented in Table 18.

Step 7. Then, the concordance matrix and discordance matrix are respectively generated as follows:

$$\mathcal{A} = \begin{bmatrix} - & 0.3333 & 0.15 & 0.55 \\ 0.7 & - & 0.0333 & 0.4666 \\ 0.8833 & 1 & - & 0.85 \\ 0.45 & 0.75 & 0.15 & - \end{bmatrix}$$

$$= [a_{\tau\kappa}]_{4\times 4}, \quad \text{and}$$

$$\mathcal{B} = \begin{bmatrix} - & 0.6434 & 0.9314 & 0.7314 \\ 0.3565 & - & 1 & 0.5719 \\ 0.0685 & 0 & - & 0.1164 \\ 0.2685 & 0.428 & 0.8718 & - \end{bmatrix} = [b_{\tau\kappa}]_{4\times 4}.$$

Step 8. The concordance threshold and discordance threshold are calculated as $\underline{\mathcal{A}} = 0.5263$ and $\underline{\mathcal{B}} = 0.4989$, respectively.

Step 9. The effective concordance matrix $\mathcal{F}$ and effective discordance matrix $\mathcal{G}$ are created as follows:

$$\mathcal{F} = \begin{bmatrix} - & 0 & 0 & 1 \\ 1 & - & 0 & 0 \\ 1 & 1 & - & 1 \\ 0 & 1 & 0 & - \end{bmatrix} = [f_{\tau\kappa}]_{4\times 4}, \quad \text{and}$$

$$\mathcal{G} = \begin{bmatrix} - & 0 & 0 & 0 \\ 1 & - & 0 & 0 \\ 1 & 1 & - & 1 \\ 1 & 1 & 0 & - \end{bmatrix} = [g_{\tau\kappa}]_{4\times 4}.$$

Step 10. Then, the aggregated outranking matrix $\mathcal{H}$ is

$$\mathcal{H} = \begin{bmatrix} - & 0 & 0 & 0 \\ 1 & - & 0 & 0 \\ 1 & 1 & - & 1 \\ 0 & 1 & 0 & - \end{bmatrix} = [h_{\tau\kappa}]_{4\times 4}.$$

Step 11. Considering the aggregated outranking matrix $\mathcal{H}$, we obtain the following binary relations as:

- $h_{21} = 1$ and $h_{12} = 0 \Rightarrow x_2 \succ x_1$,
- $h_{31} = 1$ and $h_{13} = 0 \Rightarrow x_3 \succ x_1$,
- $h_{32} = 1$ and $h_{23} = 0 \Rightarrow x_3 \succ x_2$,
- $h_{34} = 1$ and $h_{43} = 0 \Rightarrow x_3 \succ x_4$,
- $h_{42} = 1$ and $h_{24} = 0 \Rightarrow x_4 \succ x_2$.

Therefore, the ranking order of alternatives is found as $x_3 \succ x_4 \succ x_2 \succ x_1$.

7. Comparison and discussion

Algorithm 2 is more general than Algorithm 1; in other words, it is the version that takes into account parameter weights. In this section, the performances of Algorithms 2, 3, and 4 are explained and evaluated. All of these algorithms can be used to deal with multi-criteria group decision problems involving incomplete information. While each of them has a different operating philosophy, they also have one goal in common, that is, to combine the evaluations of multiple decision-makers and to propose an optimal choice. They can also offer a choice according to the assessment of only one decision-maker. While Algorithm 2 can be applied if the decision-makers determine the same impact coefficient for "undetermined", there is no such limitation for Algorithms 3 and 4. The computational performance of each of our algorithms is critically analyzed by the experimental studies; hence, we have Table 19.

As shown in Table 19, the outputs of Algorithms 3 and 4 are the same. For many decision-making problems, the results obtained from these two algorithms either are identical or overlap each other. Moreover, all of these algorithms can be used for Examples 4.3 and 4.4, and produce the same results. For the problems in

Table 19. Comparing and matching the results of Algorithms 2, 3, and 4.

Problems	Impact coefficients	Optimal choice		
		Algorithm 2	**Algorithm 3**	**Algorithm 4**
Example 4.3	$\xi = 1$ (i.e., $\xi_1 = 1$, $\xi_2 = 1$)	x_3	x_3	x_3
Example 4.4	$\xi = 1$ (i.e., $\xi_1 = 1$, $\xi_2 = 1$)	x_1	x_1	x_1
Example 5.1	$\xi_1 = 1.5$, $\xi_2 = 2$	–	x_2	x_2
Example 6.1	$\xi_1 = 2$, $\xi_2 = 3$, $\xi_3 = 2$	–	x_3	x_3

Note: "–" means that the algorithm is not applicable to this problem.

Table 20. The comparison results of Algorithms 2, 3, and 4 with some existing soft decision making algorithms.

Ref.	Problem in the paper	Optimal choice			
		Algorithm in the paper	**Algorithm 2**	**Algorithm 3**	**Algorithm 4**
[38]	Application (Section 5) in [38]	u_1	u_1	u_1	u_1
[35, 36]	(Example 5.17 in [35] Example 3.3 in [36])	h_1, h_2, h_3	h_1, h_2, h_3	h_1	$h_1 \succ h_3 \succ h_4?h_5$ & $h_2 \succ h_4?h_5$
[59]	Example 31 in [59]	u_3	u_3	u_3	u_3
[32]	Table 2 (Section 3.4) in [32]	h_1, h_6	h_1, h_6	h_1, h_6	h_1, h_6

Note: For Algorithms 2, 3, and 4, the weights of parameters in decision problem are taken equally and their sum is 1.

Examples 5.1 and 6.1, the outputs of the algorithms coincide. These results support the efficiency and usefulness of the proposed algorithms.

Since the three-valued soft set is an extension of soft set, the emerging algorithms can be applied to decision-making problems based on the soft set(s). In this respect, the results of the proposed algorithms with those of some of the existing soft decision-making algorithms were compared. The details supporting this argument are presented in Table 20.

In this table, the weights of parameters in each of these problems were equally considered when making calculations in Algorithms 2, 3, and 4 (for instance, in application (Section 5) in [38], the parameter set is $X = \{x_1, x_2, x_3\}$ and so we specify $\omega_1 = \omega_2 = \omega_3 = \frac{1}{3}$). Also, ξ can be arbitrarily chosen in accordance with the comments on the selection of the impact coefficient mentioned above. It is clear that the arbitrary selection of ξ will not change the result(s).

As shown in Table 20, the results of our algorithms coincide with those of existing soft decision-making algorithms. For Example 5.17 in [35] and Example 3.3 in [36], the optimal choice by the algorithms proposed in [35,36] is $\{h_1, h_2, h_3\}$, while the optimal choice by Algorithm 3 is h_1 (where $h_1 \succ h_3 \succ h_2$). This is not a contradiction, and this is the effect of normalizing the decision matrices in the model of TOPSIS (Algorithm 3). Considering the result of Algorithm 4 for the same problems, we say that h_1 and h_2 are incomparable, while h_3 and h_2 are incomparable. This does not contradict the result that the optimal choice is $\{h_1, h_2, h_3\}$, because what is certain is that $h_1 \succ h_4?h_5$, $h_2 \succ h_4?h_5$, and $h_3 \succ h_4?h_5$. Consequently, the applicability of our algorithms to decision making based on both soft set and three-valued soft set demonstrates their performance range and advantages.

8. Conclusion

This study defined a three-valued soft set as a generalization of the soft set and its set-theoretic operations like intersection, union, difference, and symmetric difference. Moreover, the basic relationships concerning three-valued soft sets were described and the corresponding generalization of the operations on soft sets to these sets was highlighted. In this regard, some examples for them were provided. The algorithms supporting multi-criteria decision making for the three-valued soft set based on Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) and ELimination Et Choice Translating (ELECTRE) techniques were formed and their outputs were compared. Thus, it was pointed out that these algorithms exhibited the applicability and efficiency of three-valued soft sets in handling the multi-criteria decision making involving uncertain or incomplete information.

We hope that this work will contribute to decision-making under uncertain and incomplete information in the context of soft sets and also will provide new ideas for future studies related to soft sets. Also, this study will motivate researchers to use three-valued logic stems in many practical applications such as data mining, data selection, data integration, data analysis, control of production processes, and pattern evaluation. In near future, we intend to explore new operations on three-valued soft sets and their practical applications in the fields such as science, social science, medical science, environmental science, economics, and so on.

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Biographies

Eyüp Akçetin is an Associate Professor at the Department of Accounting and Financial Management, Seydikemer School of Applied Sciences of Mugla SıtkıKocman University in Turkey. He received his MSc degrees in Logistics and Maritime Transport from Dokuz Eylül University in 2007 and received another MSc degree in Informatics and Computer Engineering from Sakarya University in 2014 Turkey. He received his PhD degrees in Maritime Economy and Maritime Business Management from Istanbul University in 2012 Turkey. His research interests include multi-criteria decision-making, game theory, business intelligence and data mining, gamification, system dynamics in business. He has contributed to these scientific topics with many articles, books, and book chapters.

Hüseyin Kamacı is a Research Assistant at the Mathematics Department in the Science and Arts Faculty of Yozgat Bozok University, Turkey. He received his MSc and PhD degrees in Mathematics from Bozok University, Yozgat, Turkey in 2014 and 2018, respectively. His research interests include mathematical logic, set theory, operational research, computational intelligence, and decision-making. He has many valuable publications on these issues in different scientific journals.